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Structured Vibration Modes of General Compound Planetary Gear Systems

This paper extends previous analytical models of simple, single-stage planetary gears to compound, multi-stage planetary gears. This model is then used to investigate the structured vibration mode and natural frequency properties of compound planetary gears of general description, including those with equally spaced planets and diametrically opposed planet pairs. The well-defined cyclic structure of simple, single-stage planetary gears is shown to be preserved in compound, multi-stage planetary gears. The vibration modes are classified into rotational, translational, and planet modes and the unique properties of each type are examined and proved for general compound planetary gears. All vibration modes fall into one of these three categories. For most cases, both the properties of the modes and the modes themselves are shown to be insensitive to relative planet positions between stages of a multi-stage system. [DOI: 10.1115/1.2345680]

1 Introduction

Planetary gears are widely used in many applications due to advantages over parallel shaft arrangements such as high power density and large reduction in a small volume [1]. In fixed-ratio applications requiring a reduction of approximately 10:1 or less, a simple (i.e., single planet for each load path), single-stage planetary gear is often sufficient. For fixed-ratio applications requiring greater reductions, however, compound planetary gears are common [2]. Automatic transmissions require more kinematic combinations than a single planetary can provide, so they also utilize compound planetary gears. Despite their benefits, Kahraman [3] notes that compound planetary gears often have more noise and vibration problems than simple planetary gears.

Although the vibration of simple, single-stage planetary gears has been studied by many researchers (e.g., [4–16]), the vibration of compound or multi-stage planetary gearsets has received little research attention. To the authors' knowledge, Kahraman [3] conducted the only analytical study of the vibration of compound planetary gears. In that work, a purely rotational model that does not include gear translation is applied to a restricted class of compound planetary gears, and modal properties are given based on simulation rather than derivation.

For simple, single-stage planetaries, Lin and Parker [4] analyzed the free vibration and proved that there are exactly three types of modes: Rotational, translational, and planet. They showed certain unique properties for these modes. These findings are consistent with finite element results [8]. Recent industry-motivated analyses by the authors investigated a variety of compound and multi-stage planetary gearsets and found that they all exhibited modal characteristics similar to a simple, single-stage planetary gear. The intent of this work is to generalize prior results and mathematically prove that compound, multi-stage planetary gear systems of general description possess highly structured modal properties analogous to simple, single-stage planetary gears. This work includes all of the cases considered in Kahraman's model [3]. It expands on that work by allowing an infinite number of kinematic combinations instead of a limited number, including translational degrees of freedom in addition to rotations, and providing analytical proof of the modal properties in addition to numerical verification.

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2 Modeling and Equations of Motion

There are many different kinds of gear sets that are referred to as compound planetary gears. The present work attempts to be as general as possible in its definition. Three different types of compound planetary gears are described. The first two are the *stepped planet* planetary (Fig. 1), and the *meshed planet* planetary (Fig. 2) [17]. The third type, the multi-stage planetary, (Fig. 3) is created by joining multiple planetary stages together, where each stage can be a simple, meshed, or stepped planetary. No limitations are imposed on which elements are the input or output members.

In deriving the equations of motion, it is necessary to systematically enumerate the three types of elements in a planetary gear: Carriers, central gears (suns and rings), and planets. Let the carriers be numbered $1,2,\ldots,a$ and the central gears be numbered $1,2,\ldots,b$. In order to describe the planets, it is convenient to divide them into groups.

First, each planet is always associated with one and only one carrier. Thus, it is natural to define a *planet set* to be all of the planets associated with a particular carrier. Within each planet set, the planets are partitioned into isolated *planet trains*. Two planets are considered to be in the same planet train if they: (a) Mesh with each other, or (b) are connected to each other by a shaft (as in stepped planets). The planet train concept is illustrated in Figs. 1 and 2.

Let the planet sets be numbered $1,2,\ldots,a$, where planet set i is associated with carrier i. Let the planet trains be numbered $1,2,\ldots,c^i$, where c^i designates the number of planet trains in planet set i. Let the planets in a train be numbered $1,2,\ldots,d^i$, where d^i designates the number of planets per planet train in planet set i. The model requires all planet trains in a planet set to have the same number of planets, which is generally the case in practical systems.

2.1 Coordinates and Geometric Description. A planar problem is considered where each carrier, planet, and central gear has three degrees of freedom: Two translational and one rotational. This limits the analysis to spur gears. In total, the model has $3(a+b+\sum_{i=1}^{a}c^i d^i)$ degrees of freedom.

The choice of a coordinate basis for the multi-stage case presents some options. For single stage planetaries, Lin and Parker [4] chose a basis that rotates with the carrier. This allows the planet positions to be described by fixed angles even with gear rotation. In the present case, there are potentially several different

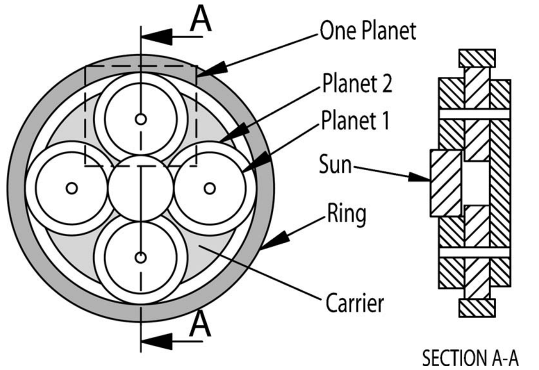

Fig. 1 Example of a stepped planet compound planetary gear

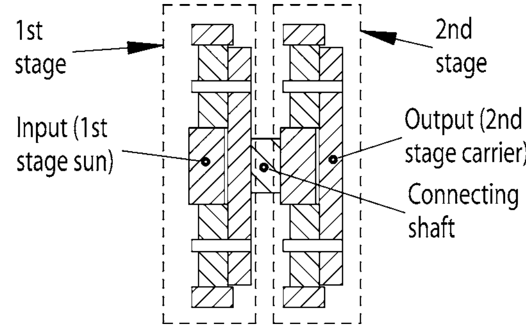

Fig. 3 Example of a multi-stage compound planetary gear

carriers, each rotating at different speeds. It is possible to construct multiple rotating bases and then describe the coupling between the rotating components by means of time-varying coordinate transformations. Another possibility is to describe all components in a single fixed basis and then allow the planet position angles to change with time. Either choice leads to a time-varying stiffness matrix under operating conditions. For the present work considering free vibration, the second choice is more convenient.

Using a fixed basis, translational coordinates x_c^i, y_c^i are assigned to the carriers and translational coordinates x_g^j, y_g^j are assigned to the central gears (suns and rings). The x_c^i and x_g^j coordinates are chosen to be positive towards the equilibrium position of the arbitrarily chosen first planet of the first planet train of the first planet set at time $t=0$. Translational coordinates ζ_p^{ilm}, η_p^{ilm} are assigned to planet m in planet train l in planet set i. These are radial and tangential coordinates, respectively (Fig. 4). The origin of the ζ_p^{ilm}, η_p^{ilm} coordinates is fixed at each planet's equilibrium position and does not translate with carrier vibration, that is, ζ_p^{ilm} and η_p^{ilm} are *absolute* (not relative) deflections.

These coordinates illustrate conventions that are followed throughout this paper. For all variables, subscripts denote labels and superscripts denote indices. Indices denoting the gear and carrier (if any) always precede indices for the planet set, planet train, and planet (if any). Indices denoting the planet set, planet train, and planet are always in that order. Not all of these indices will be present for every variable. In general, i is used for denoting carriers and planet sets, j for central gears, l for planet trains, and m for planets. Some expressions require more than one of a particular component type (e.g., two central gears). In that case an additional index (either n or f) will be used for the second component. Which component the second index represents can be determined from context. For example, in m_p^{ilm}, the indices are planet set, planet train, and planet while in $k_{g,xx}^{jn}$ the indices are both gears.

In [4], all rotational coordinates are chosen to be $u=r\theta$, where r is the base radius for gears and the radius from the carrier center to the planet center for carriers. In the present work, this form cannot be used for the carrier because the meshed planet case has planets at multiple radii (e.g., Fig. 2). Thus, the carrier rotational coordinate must be θ. It is then more convenient to use θ instead of u for central gears as well. Due to this difference, the matrices $\mathbf{K}_{c1}$, $\mathbf{K}_{g1}$, $\mathbf{K}_{cp}$, and $\mathbf{K}_{gp}$ given in Appendix C differ from the corresponding matrices in [4]. For the planets, the choice is arbitrary. u is used because it simplifies the notation. Therefore, corresponding planet matrices are the same as in [4].

The circumferential planet locations are specified by the time-varying (under operating conditions) angles $\psi^{ilm}(t)$, where $\psi^{111}(0)=0$. To describe the orientation of meshed planets with respect to each other, the angle β^{ilmn} is introduced. This angle is measured counterclockwise from the positive ζ direction of planet m to a line connecting the centers of planets m and n (both planets are in planet set i, planet train l). This is illustrated in Fig. 4, where both β^{ilmn} and β^{ilnm} are shown.

2.2 Equations of Motion. The equations of motion are similar to [4] except there are more elements and they can be coupled in additional ways. Gyroscopic effects are neglected. The equa-

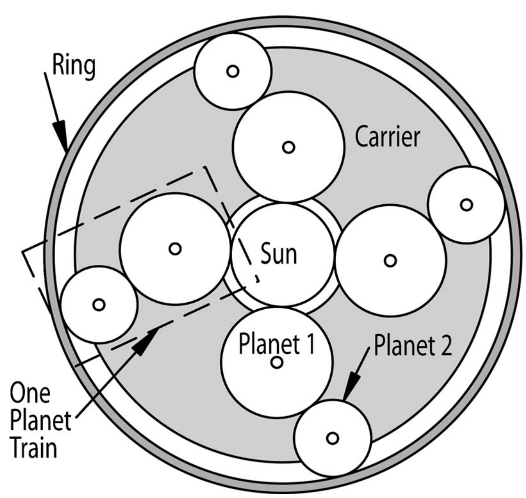

Fig. 2 Example of a meshed planet compound planetary gear

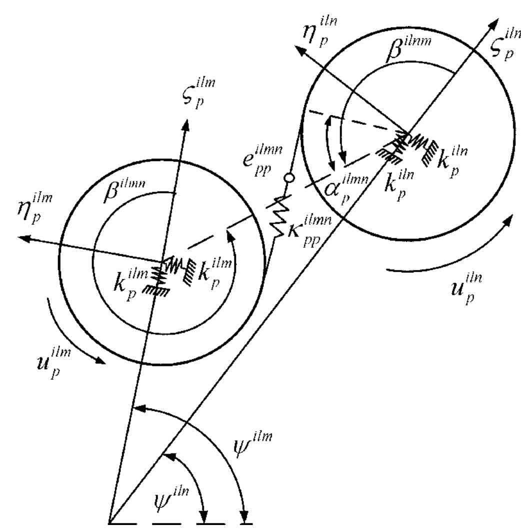

Fig. 4 A planet-planet mesh modeled by a linear spring and static transmission error

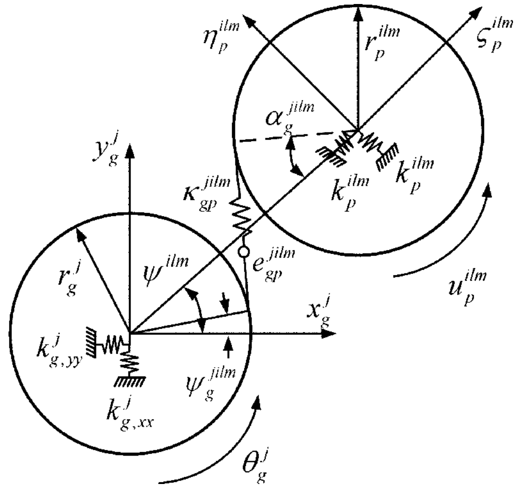

Fig. 5 A sun-planet mesh modeled by a linear spring and static transmission error

tions of motion of a central gear are derived as an example. The forces/moments on a central gear fall into five categories: Gear-gear forces/moments, gear-carrier forces/moments, gear-ground bearing forces/moments, gear-planet mesh forces, and externally applied forces/moments. The first three are easily described. Modeling the interaction as a linear spring, such as from a connecting shaft, the gear-gear force between gears j and n in the x direction is described by $k_{g,xx}^{jn}(x_g^j - x_g^n)$. Similar terms occur for the y and θ coordinates as well as the gear-carrier and gear-ground forces/moments. Note that central gears never mesh with other central gears.

Gear-planet mesh forces are more complicated to describe. The gear-planet mesh between gear j and planet m of planet train l, planet set i is shown in Fig. 5. This is nearly identical to Fig. 2 in [4], except that all elements are enumerated with additional subscripts and superscripts. For clarity, κ indicates a tooth mesh stiffness, K indicates a torsional shaft stiffness, and k indicates a translational bearing or shaft stiffness. Although a sun is shown in Fig. 5, the situation for a ring is similar. The equations for a ring and a sun differ only in the sign of a few terms. In order to generalize these equations, the variable σ^j is introduced as

$$\sigma^j = \begin{cases} 1 & \text{if gear } j \text{ is a ring (internal gear)} \\ -1 & \text{if gear } j \text{ is a sun (external gear)} \end{cases}$$

In order to represent the changing number of teeth in contact, the gear-planet mesh is modeled as a time-varying mesh stiffness $\kappa_{gp}^{jilm}(t)$. The deflection of the tooth mesh is

$$\delta_g^{jilm} = y_g^j \cos \psi_g^{jilm}(t) - x_g^j \sin \psi_g^{jilm}(t) + r_g^j \theta_g^j + \sigma^j \zeta_p^{ilm} \sin \alpha_g^{jilm} - \eta_p^{ilm} \cos \alpha_g^{jilm} - \sigma^j u_p^{ilm} + e_{gp}^{jilm}(t)$$

$$\psi_g^{jilm}(t) = \psi^{ilm}(t) + \sigma^j \alpha_g^{jilm} \tag{1}$$

where α_g^{jilm} is the pressure angle of the mesh, and $e_{gp}^{jilm}(t)$ is the time-varying, unloaded static transmission error (compression is positive), which can be used to model excitation due to profile modifications and manufacturing errors.

Denoting externally applied forces and torques by F_g^j, τ_g^j, respectively, and mass and mass moment of inertia by I_g^j, m_g^j, respectively, the equations of motion for gear j are

$$m_g^j \ddot{x}_g^j + k_{gb,xx}^j x_g^j - \sum_{i=1}^{a} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} \sin \psi_g^{jilm}(t) + \sum_{i=1}^{a} k_{cg,xx}^{ij} \times (x_g^j - x_c^i) + \sum_{n=1,n\neq j}^{b} k_{g,xx}^{jn}(x_g^j - x_g^n) = F_{g,xx}^j(t)$$

$$m_g^j \ddot{y}_g^j + k_{gb,yy}^j y_g^j + \sum_{i=1}^{a} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} \cos \psi_g^{jilm}(t) + \sum_{i=1}^{a} k_{cg,yy}^{ij} \times (y_g^j - y_c^i) + \sum_{n=1,n\neq j}^{b} k_{g,yy}^{jn}(y_g^j - y_g^n) = F_{g,yy}^j(t)$$

$$I_g^j \ddot{\theta}_g^j + K_{gb,\theta\theta}^j \theta_g^j + \sum_{i=1}^{a} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} r_g^j + \sum_{i=1}^{a} K_{cg,\theta\theta}^{ij}(\theta_g^j - \theta_c^i) + \sum_{n=1,n\neq j}^{b} K_{g,\theta\theta}^{jn}(\theta_g^j - \theta_g^n) = \tau_g^j(t) \tag{2}$$

where all symbols are defined in the Nomenclature section.

The equations of motion for the planets (see Fig. 4) and the carriers are similarly obtained. These are given in Appendix A. For notational convenience, the bearings between the planets and the carriers are assumed to be isotropic (i.e., have the same stiffness in all directions).

The equations of motion for the system are written in matrix form as

$$\mathbf{M}\ddot{\mathbf{x}}(t) + [\mathbf{K}_b + \mathbf{K}_m(t)]\mathbf{x}(t) = \mathbf{F}(t) \tag{3}$$

where $\mathbf{M}$ is the positive definite mass matrix, $\mathbf{K}_\mathrm{b}$ is the diagonal bearing stiffness matrix, $\mathbf{K}_m(t)$ is the symmetric stiffness matrix from coupling between elements (both tooth meshes and shaft couplings), and $\mathbf{F}$(t) is the vector of applied forces and torques. The vector $\mathbf{x}$ and matrix components are given in Appendix B.

3 Natural Frequencies and Vibration Modes

To determine the natural frequencies and vibration modes the time-invariant system is considered. All mesh stiffnesses are considered to be constant and equal to their average stiffness over one mesh cycle. The planet position angles $\psi^{ilm}(t)$ are fixed at their values for some arbitrary time. All externally applied forces/moments are assumed to be zero.

The associated eigenvalue problem derived from $\mathbf{x}(t) = \mathbf{q}e^{i\omega t}$ is

$$\omega^2 \mathbf{M}\mathbf{q} = (\mathbf{K}_b + \mathbf{K}_m)\mathbf{q} \tag{4}$$

$$\mathbf{q} = \left[\mathbf{q}_c^1 \ \cdots \ \mathbf{q}_c^a | \mathbf{q}_g^1 \ \cdots \ \mathbf{q}_g^b | \mathbf{q}_{ps}^1 \ \cdots \ \mathbf{q}_{ps}^a\right]^T \tag{5}$$

The individual vectors for the carriers and gears are

$$\mathbf{q}_c^i = [x_c^i \ y_c^i \ \theta_c^i]^T, \quad \mathbf{q}_g^j = \left[x_g^j \ y_g^j \ \theta_g^j\right]^T \tag{6}$$

The planet set, planet train, and planet vectors are

$$\mathbf{q}_{ps}^i = [\mathbf{q}_{pt}^{i1} \ \cdots \ \mathbf{q}_{pt}^{ic^i}]^T, \quad \mathbf{q}_{pt}^{il} = [\mathbf{q}_p^{il1} \ \cdots \ \mathbf{q}_p^{ild^i}]^T,$$

$$\mathbf{q}_p^{ilm} = [\zeta_p^{ilm} \ \eta_p^{ilm} \ u_p^{ilm}]^T \tag{7}$$

Expanding (4) into three groups of equations for the individual components according to the matrix definitions in Appendix B yields

$$(\mathbf{K}_{cb}^i - \omega^2 \mathbf{M}_c^i)\mathbf{q}_c^i + \sum_{f=1}^{a} \mathbf{K}_c^{if} \mathbf{q}_c^f + \sum_{j=1}^{b} \mathbf{K}_{cg}^{ij} \mathbf{q}_g^j + \mathbf{K}_{c,ps}^i \mathbf{q}_{ps}^i = 0$$

$$i = 1,2, \ldots ,a \tag{8}$$

Table 1 Parameters of example system in Fig. 6

Number of Carriers	$a=2$
Number of Gears	$b=3$
Number of Planet Trains	$c^1=4,\ c^2=5$
Number of Planets per Train	$d^1=3,\ d^2=1$
Pressure Angles	$\alpha_g^{jilm}=\alpha_p^{jilm}=25°$
Planet Location	$\psi^{1l1}(0)=\frac{2\pi(l-1)}{4},\ \psi^{1l2}(0)=\psi^{1l3}(0)=\psi^{1l1}(0)+32°,$ $\psi^{2l1}(0)=\frac{2\pi(l-1)}{5},\ \beta^{1l12}=70°,\ \beta^{1l21}=218°$
Mesh Stiffnesses (N/m)	$\kappa_{pp}^{ilmn}=\begin{cases}5\times10^8 & \text{if } i=1,m=1,n=2 \text{ or } i=1,m=2,n=1\\ 0 & \text{otherwise}\end{cases}$ $\kappa_{gp}^{jilm}=\begin{cases}5\times10^8 & \text{if } j=1,i=1,m=1\\ 5\times10^8 & \text{if } j=2,i=1,m=3\\ 5\times10^8 & \text{if } i=2 \text{ and } (j=2 \text{ or } j=3)\\ 0 & \text{otherwise}\end{cases}$
Translational Bearing Stiffnesses (N/m)	$k_{cb,xx}^i=k_{cb,yy}^i=1\times10^9,\ k_{gb,xx}^j=k_{gb,yy}^j=1\times10^9,\ k_p^{ilm}=1\times10^9$
Torsional Reaction Stiffnesses (N-m/rad)	$K_{cb,\theta\theta}^i=0,\ K_{gb,\theta\theta}^1=K_{gb,\theta\theta}^3=0,\ K_{gb,\theta\theta}^2=5\times10^8$
Translation Coupling Stiffness (N/m)	$k_{g,xx}^{jn}=k_{g,yy}^{jn}=0,\ k_{c,xx}^{if}=k_{c,yy}^{if}=0,$ $k_{cg,xx}^{ij}=k_{cg,yy}^{ij}=\begin{cases}8\times10^8 & \text{if } i=1,j=3\\ 0 & \text{otherwise}\end{cases}$ $k_{pp,\zeta\zeta}^{ilmn}=k_{pp,\eta\eta}^{ilmn}=\begin{cases}1\times10^9 & \text{if } i=1,m=2,n=3 \text{ or } i=1,m=3,n=2\\ 0 & \text{otherwise}\end{cases}$
Torsional Coupling Stiffness (N-m/rad)	$K_{g,\theta\theta}^{jn}=0,\ K_{c,\theta\theta}^{ih}=0,\ K_{cg,\theta\theta}^{ij}=\begin{cases}2\times10^8 & \text{if } i=1,j=3\\ 0 & \text{otherwise}\end{cases}$ $k_{pp,uu}^{ilmn}=\begin{cases}1\times10^7 & \text{if } i=1,m=2,n=3 \text{ or } i=1,m=3,n=2\\ 0 & \text{otherwise}\end{cases}$
Mass (kg)	$m_g^1=m_g^3=1, m_g^2=5, m_p^{1l1}=m_p^{1l2}=m_p^{1l3}=0.75$ $m_p^{2l1}=2, m_c^i=10$
Mass Moment of Inertia (kg-m^2)	$I_g^1=I_g^3=0.05, I_g^2=0.5, I_c^1=1.0, I_c^2=1.2$ $I_p^{1l1}=I_p^{1l2}=0.01, I_p^{1l3}=0.09, I_p^{2l1}=0.2$
Radii (mm)	$r_g^1=r_g^3=100, r_g^2=300, r_p^{1l1}=60, r_p^{1l2}=77.7, r_p^{1l3}=55.8$ $r_p^{2l1}=100, r_c^{1l1}=176.5, r_c^{1l2}=r_c^{1l3}=270, r_c^{2l1}=220.7$
Gear Types	$\sigma^1=\sigma^3=-1$ (external, sun) , $\sigma^2=1$ (internal, ring)

$$(\mathbf{K}_{gb}^j-\omega^2\mathbf{M}_g^j)\mathbf{q}_g^j+\sum_{n=1}^{b}\mathbf{K}_g^{jn}\mathbf{q}_g^n+\sum_{i=1}^{a}(\mathbf{K}_{cg}^{ij})^T\mathbf{q}_c^i+\sum_{i=1}^{a}\mathbf{K}_{g,ps}^{ji}\mathbf{q}_{ps}^i=0$$

$$j=1,2,\ldots,b \tag{9}$$

$$(\mathbf{K}_{ps}^i-\omega^2\mathbf{M}_{ps}^i)\mathbf{q}_{ps}^i+(\mathbf{K}_{c,ps}^i)^T\mathbf{q}_c^i+\sum_{j=1}^{b}(\mathbf{K}_{g,ps}^{ji})^T\mathbf{q}_g^j=0 \quad i=1,2,\ldots,a \tag{10}$$

The following additional assumptions are imposed:

(1) All planet trains within a planet set are identical in all ways (mass, tooth parameters, bearing properties, etc.);
(2) All planet trains are equally spaced around their associated carrier. The case with diametrically opposed pairs of planet trains is discussed later;
(3) For each planet set there are three or more planet trains ($c^i \geqslant 3$). In practice, almost all planetary gears have three or more planets to take advantage of load sharing. The case with two equally spaced planet trains is considered later as a case of diametrically opposed planet trains;
(4) All bearing and shaft stiffnesses are isotropic.

These assumptions lead to a cyclically symmetric structure with distinctive vibration properties. These properties are first illustrated by an example using the parameters in Table 1. This system is shown schematically in Fig. 6. It is a two stage system where the first stage has both meshed and stepped planets and the second stage has only simple planets. The ring gear is common to both

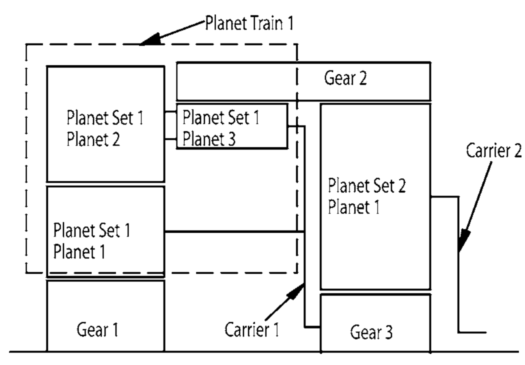

Fig. 6 Example system

Table 2 Natural frequencies and their multiplicities m for the example system. Rotational, translational, and planet modes are distinguished by R, T, and P, respectively.

Natural Frequency (Hz)	
m=1	m=2
0.0 R	442.2 T
660.9 P	771.3 T
932.0 R	1029.5 P
1011.2 R	1436.5 T
1261.4 R	1856.2 T
1557.1 R	2250.2 T
2382.2 P	3499.2 T
3379.0 R	3890.2 P
3796.3 R	4491.2 T
4849.2 R	4802.9 P
5811.5 P	5503.2 T
5860.6 R	5553.4 T
5991.1 P	5936.4 T
6019.5 R	6141.3 T
6247.8 P	7796.3 T
6597.2 R	8360.2 T
6981.1 R	8812.4 T
7180.6 P	9117.1 T
7767.1 R	10083.9 T
8339.6 P	11071.6 T
8564.8 R	
10082.7 P	
10085.6 R	
10982.1 R	
11054.3 P	
11092.6 R	

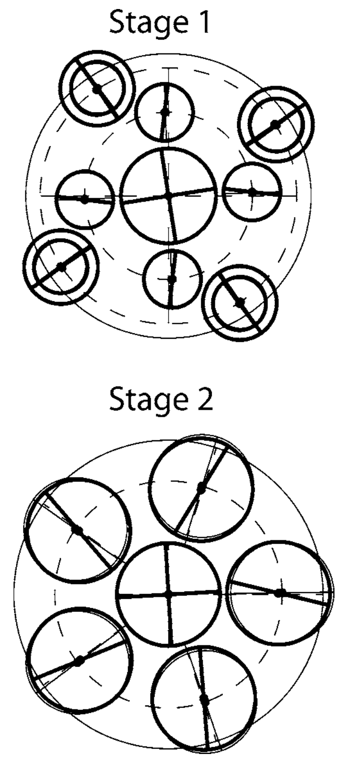

Fig. 7 Typical rotational mode for example system of Fig. 6 and Table 1, ω=932.0 Hz

stages. The system has 66 degrees of freedom. The natural frequencies and their multiplicities are shown in Table 2.

All vibration modes for this system can be classified into one of three types. Typical vibration modes of each type are shown in Figs. 7–9. The equilibrium positions of the gears and carriers are shown as dashed lines. The equilibrium positions of the planets with respect to the displaced carriers are shown as light lines. The displaced positions of the suns and planets are shown as heavy lines. Dots represent the component centers. Motion of the ring is omitted for clarity.

Figure 7 illustrates a type of mode where all central gears and carriers have pure rotation and no translation. These are named *rotational modes*. In a rotational mode, all planet trains within a given planet set have identical motion. There are exactly $a+b+3\Sigma_{i=1}^{a}d^i=17$ rotational modes, each with an associated natural frequency of multiplicity one (distinct).

Figure 8 shows a pair of degenerate modes that have the same natural frequency. All central gears and carriers have pure translational motion with no rotation. These are called *translational modes*. There are exactly $a+b+3\Sigma_{i=1}^{a}d^i=17$ degenerate pairs of translational modes, where each pair has an associated natural frequency of multiplicity two. Well-defined relations between the planet motions will be shown.

Figure 9 illustrates two modes where the carriers and central gears have no motion; the planets are the only components that deflect. These are called *planet modes*. A given mode is associated with motion of the planets of exactly one planet set and planets in all other planet sets have no motion. In general, each natural frequency associated with planet set i has multiplicity c^i-3. Thus, planet modes exist only for planet sets containing four or more planet trains. If planet modes exist for planet set i, then there are $3d^i$ different natural frequencies for that set's planet modes (each with multiplicity c^i-3). So, there are exactly $3\Sigma_{i=1}^{a}(c^i-3)d^i=15$ planet modes. The number of different natural frequencies for a planet set's planet modes is dictated by the number of degrees of freedom in one planet train; their multiplicity is dictated by the number of trains in the set. Finally, each planet train's motions are a scalar multiple of the first (or any arbitrarily chosen) planet train's motions. Equations governing these scalars will be derived.

The above properties of the modes, which have been drawn from numerical results, are now proven analytically. The proof proceeds by proposing candidate modes based on the numerical results and substituting them into the equations of motion. For each type of mode, a reduced degree of freedom eigenvalue problem is found. The total number of eigenvalues is shown to equal the total number of degrees of freedom in the system, so the three mode types are an exhaustive list of the possible mode types.

The equations of motion contain many sums over the planets and planet trains, and many of these can be simplified with the numbered assumptions noted above. Assumption 1 leads directly to the simplifications

$$\mathbf{M}_{pt}^{il}=\mathbf{M}_{pt}^{i1}, \quad \mathbf{K}_{pt}^{il}=\mathbf{K}_{pt}^{i1} \text{ for all } i,l \tag{11}$$

Assumption 2 is stated formally as $\psi^{j(l+1)m}=2\pi/c^i+\psi^{jlm}$ for all i,l,m. Together with assumption 3, this implies the relations

$$\sum_{l=1}^{c^i}\sin\psi^{jlm}=0, \quad \sum_{l=1}^{c^i}\cos\psi^{jlm}=0 \tag{12}$$

$$\sum_{l=1}^{c^i}\cos\psi^{jlm}\sin\psi^{jlm}=0 \tag{13}$$

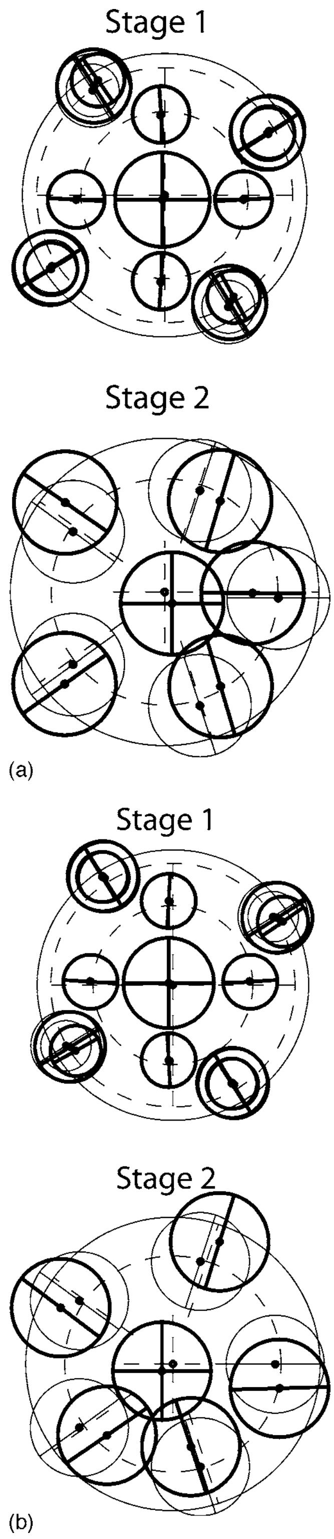

Fig. 8 A pair of typical translational modes for example system of Fig. 6 and Table 1, ω=3499.2 Hz

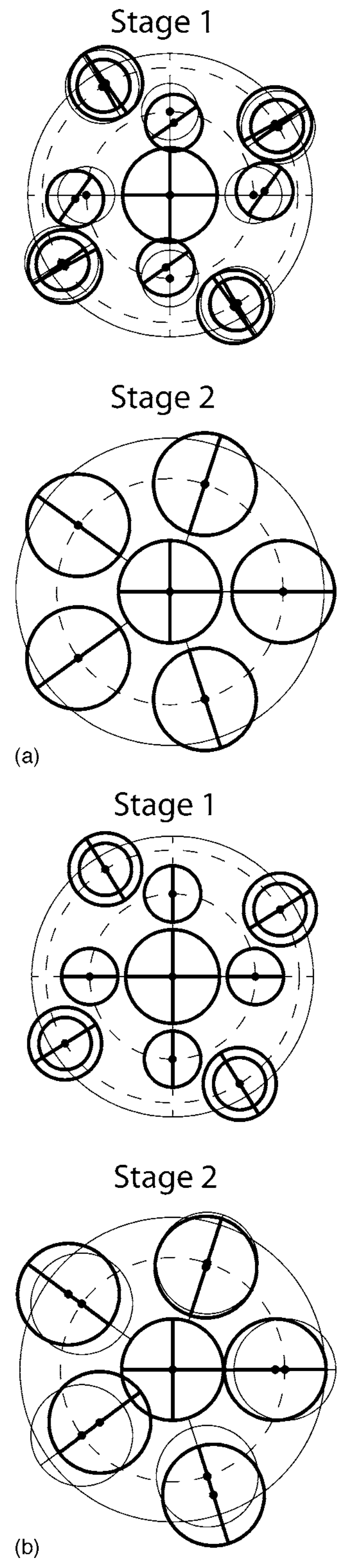

Fig. 9 Two typical planet modes for the example system of Fig. 6 and Table 1. (*a*) A mode in which stage 1 planets have motion and stage 2 has no motion, ω=2382.2 Hz (*b*) A mode in which stage 1 has no motion and stage 2 planets have motion, ω=3890.2 Hz.

$$\sum_{l=1}^{c^i} \cos \psi^{jlm} \sin \hat{\psi}^{jlm} = -\sum_{l=1}^{c^i} \cos \hat{\psi}^{jlm} \sin \psi^{jlm} \tag{14}$$

$$\sum_{l=1}^{c^i} \cos^2 \psi^{jlm} = \sum_{l=1}^{c^i} \sin^2 \psi^{jlm} \tag{15}$$

$$\sum_{l=1}^{c^i} \cos \psi^{jlm} \cos \hat{\psi}^{jlm} = \sum_{l=1}^{c^i} \sin \psi^{jlm} \sin \hat{\psi}^{jlm} \tag{16}$$

where $\hat{\psi}^{jlm}=\psi^{jlm}-\psi^{j1m}$ so that $\hat{\psi}^{j1m}=0$ for all i,l,m.

Because Assumption 1 implies $\alpha_g^{jilm}=\alpha_g^{ji1m}$ for all l, (12)–(16) also hold if ψ^{jilm} is replaced with ψ_g^{jilm}, where $\psi_g^{jilm}=\psi^{jilm}+\sigma^j\alpha_g^{jilm}$. Specifically,

$$\sum_{l=1}^{c^i} \sin \psi_g^{jilm}=0, \quad \sum_{l=1}^{c^i} \cos \psi_g^{jilm}=0 \tag{17}$$

$$\sum_{l=1}^{c^i} \cos \psi_g^{jilm} \sin \psi_g^{jilm}=0 \tag{18}$$

$$\sum_{l=1}^{c^i} \cos \psi_g^{jilm} \sin \hat{\psi}^{jilm}=-\sum_{l=1}^{c^i} \cos \hat{\psi}^{jilm} \sin \psi_g^{jilm} \tag{19}$$

$$\sum_{l=1}^{c^i} \cos^2 \psi_g^{jilm}=\sum_{l=1}^{c^i} \sin^2 \psi_g^{jilm} \tag{20}$$

$$\sum_{l=1}^{c^i} \cos \psi_g^{jilm} \cos \hat{\psi}^{jilm}=\sum_{l=1}^{c^i} \sin \psi_g^{jilm} \sin \hat{\psi}^{jilm} \tag{21}$$

where $\hat{\psi}_g^{jilm}=\psi_g^{jilm}-\psi_g^{ji1m}$.

3.1 Rotational Modes. A candidate rotational mode of the form (5)–(7) is given by

$$\mathbf{q}_c^i=[0\ 0\ \theta_c^i]^T \quad \mathbf{q}_g^j=[0\ 0\ \theta_g^j]^T$$

$$\mathbf{q}_{ps}^i=[\mathbf{q}_{pt}^{i1}\ \mathbf{q}_{pt}^{i1}\ \cdots\ \mathbf{q}_{pt}^{i1}]^T \quad \begin{matrix} i=1,2,\ldots,a \\ j=1,2,\ldots,b \end{matrix} \tag{22}$$

This candidate mode must satisfy the eigenvalue problem of (8)–(10). Insertion of (22) into (8) and use of (12) and Assumption 1 yields a equations (the other $2a$ equations are identically satisfied) that simplify to

$$\begin{aligned} &K_{cb,\theta\theta}^i\theta_c^i-\omega^2 I_c^i\theta_c^i+c^i\sum_{m=1}^{d^i}(r_c^{i1m})^2k_p^{i1m}\theta_c^i+\sum_{f=1,f\neq i}^{a}K_{c,\theta\theta}^{if}(\theta_c^i-\theta_c^f)+ \\ &\sum_{j=1}^{b}K_{cg,\theta\theta}^{ij}(\theta_c^i-\theta_g^j)-c^i\sum_{m=1}^{d^i}r_c^{i1m}k_p^{i1m}\eta_p^{i1m}=0 \\ &\qquad i=1,2,\ldots,a \end{aligned} \tag{23}$$

Insertion of (22) into (9) and use of (17) and Assumption 1 yields b equations

$$\begin{aligned} &K_{gb,\theta\theta}^j\theta_g^j-\omega^2 I_g^j\theta_g^j+\sum_{i=1}^{a}K_{cg,\theta\theta}^{ij}(\theta_g^j-\theta_c^i)+\sum_{n=1,n\neq j}^{b}K_{g,\theta\theta}^{jn}(\theta_g^j-\theta_g^n) \\ &\quad +r_g^j\sum_{i=1}^{a}c^i\sum_{m=1}^{d^i}\kappa_{gp}^{ji1m}(r_g^j\theta_g^j+\sigma^i\zeta_p^{i1m}\sin\alpha_g^{ji1m}-\eta_p^{i1m}\cos\alpha_g^{ji1m} \\ &\quad -\sigma^j u_p^{i1m})=0, \quad j=1,2,\ldots,b \end{aligned} \tag{24}$$

Insertion of (22) into (10) and use of (11) yields

$$\begin{aligned} &\left\{\begin{bmatrix}\mathbf{K}_{pt}^{i1} & & & \\ & \mathbf{K}_{pt}^{i1} & & \\ & & \ddots & \\ & & & \mathbf{K}_{pt}^{i1}\end{bmatrix}-\omega^2\begin{bmatrix}\mathbf{M}_{pt}^{i1} & & & \\ & \mathbf{M}_{pt}^{i1} & & \\ & & \ddots & \\ & & & \mathbf{M}_{pt}^{i1}\end{bmatrix}\right\}\begin{bmatrix}\mathbf{q}_{pt}^{i1} \\ \mathbf{q}_{pt}^{i1} \\ \vdots \\ \mathbf{q}_{pt}^{i1}\end{bmatrix} \\ &\quad +\begin{bmatrix}(\mathbf{K}_{c,pt}^{i1})^T \\ (\mathbf{K}_{c,pt}^{i2})^T \\ \vdots \\ (\mathbf{K}_{c,pt}^{ic^i})^T\end{bmatrix}\begin{bmatrix}0 \\ 0 \\ \theta_c^i\end{bmatrix}+\sum_{j=1}^{b}\begin{bmatrix}(\mathbf{K}_{g,pt}^{ji1})^T \\ (\mathbf{K}_{g,pt}^{ji2})^T \\ \vdots \\ (\mathbf{K}_{g,pt}^{jic^i})^T\end{bmatrix}\begin{bmatrix}0 \\ 0 \\ \theta_g^j\end{bmatrix}=0, \\ &\qquad i=1,2,\ldots,a \end{aligned} \tag{25}$$

By Assumption 1, the product in the next to last term in (25) is the same for every planet train, that is, the following product is independent of l for arbitrary i:

$$\begin{aligned}(\mathbf{K}_{c,pt}^{il})^T\begin{bmatrix}0 \\ 0 \\ \theta_c^i\end{bmatrix}&=[0\ -r_c^{i11}\theta_c^i\ 0\ 0\ -r_c^{i12}\theta_c^i\ 0\ \cdots\ 0 \\ &\quad -r_c^{i1d^i}\ \theta_c^i\ 0]^T \text{ for all } i,l\end{aligned}$$

A similar result holds for the last term in (25). Thus, for each i, (25) consists of c^i sets of identical equations and thus can be represented by the single independent set for the first planet train. Specifically,

$$\begin{aligned} &\left\{\begin{bmatrix}\mathbf{K}_p^{i111} & \mathbf{K}_p^{i112} & \cdots & \mathbf{K}_p^{i11d^i} \\ \mathbf{K}_p^{i121} & \mathbf{K}_p^{i122} & & \vdots \\ \vdots & & \ddots & \vdots \\ \mathbf{K}_p^{i1d^i1} & \cdots & \cdots & \mathbf{K}_p^{i1d^id^i}\end{bmatrix}\right. \\ &\quad \left.-\omega^2\begin{bmatrix}\mathbf{M}_p^{i111} & & & \\ & \mathbf{M}_p^{i122} & & \\ & & \ddots & \\ & & & \mathbf{M}_p^{i1d^id^i}\end{bmatrix}\right\}\begin{bmatrix}\mathbf{q}_p^{i11} \\ \mathbf{q}_p^{i12} \\ \vdots \\ \mathbf{q}_p^{i1d^i}\end{bmatrix}+\begin{bmatrix}(\mathbf{K}_{cp}^{i11})^T \\ (\mathbf{K}_{cp}^{i12})^T \\ \vdots \\ (\mathbf{K}_{cp}^{i1d^i})^T\end{bmatrix} \\ &\quad \times\begin{bmatrix}0 \\ 0 \\ \theta_c^i\end{bmatrix}+\sum_{j=1}^{b}\begin{bmatrix}(\mathbf{K}_{gp}^{ji11})^T \\ (\mathbf{K}_{gp}^{ji12})^T \\ \vdots \\ (\mathbf{K}_{gp}^{ji1d^i})^T\end{bmatrix}\begin{bmatrix}0 \\ 0 \\ \theta_g^j\end{bmatrix}=0, \quad i=1,2,\ldots,a \end{aligned} \tag{26}$$

Equations (23), (24), and (26) consist of $a+b+3\Sigma_{i=1}^a d^i$ linear, homogenous equations for $a+b+3\Sigma_{i=1}^a d^i$ unknowns and the parameter ω^2. Thus (23), (24), and (26) constitute a reduced degree of freedom eigenvalue problem yielding $a+b+3\Sigma_{i=1}^a d^i$ natural frequencies and rotational modes. From the solutions of the reduced eigenvalue problem, rotational vibration modes of the full system are constructed according to (22). In the general case these eigenvalues are distinct. In certain cases, however, parameters may be such that two or more eigenvalues coincide. For example, consider the example presented in the last section (Fig. 6 and Table 1) but allow the ring gear to spin freely (i.e., $k_{gb,\theta\theta}^2=0$). In this case, there are two different rotational rigid body modes ($\omega=0$).

3.2 Translational Modes. A candidate pair of translational modes of the form (5)–(7) with degenerate natural frequency ω is given by

$$\mathbf{q}=[\mathbf{q}_c^1\ \cdots\ \mathbf{q}_c^a|\mathbf{q}_g^1\ \cdots\ \mathbf{q}_g^b|\mathbf{q}_{ps}^1\ \cdots\ \mathbf{q}_{ps}^a]^T \tag{27}$$

$$\bar{\mathbf{q}}=[\bar{\mathbf{q}}_c^1\ \cdots\ \bar{\mathbf{q}}_c^a|\bar{\mathbf{q}}_g^1\ \cdots\ \bar{\mathbf{q}}_g^b|\bar{\mathbf{q}}_{ps}^1\ \cdots\ \bar{\mathbf{q}}_{ps}^a]^T \tag{28}$$

(the overbar does *not* denote complex conjugate). These are orthogonal with respect to the mass matrix such that $\bar{\mathbf{q}}^T\mathbf{M}\mathbf{q}=0$. The carrier, gear, and planet translations are related by[1]

$$\mathbf{q}_c^i = [x_c^i \; y_c^i \; 0]^T, \quad \bar{\mathbf{q}}_c^i = [y_c^i \; -x_c^i \; 0]^T \tag{29}$$

$$\mathbf{q}_g^j = [x_g^j \; y_g^j \; 0]^T, \quad \bar{\mathbf{q}}_g^j = [y_g^j \; -x_g^j \; 0]^T \tag{30}$$

$$\begin{bmatrix} \mathbf{q}_p^{ilm} \\ \bar{\mathbf{q}}_p^{ilm} \end{bmatrix} = \begin{bmatrix} \mathbf{C}^{ilm} & \mathbf{S}^{ilm} \\ -\mathbf{S}^{ilm} & \mathbf{C}^{ilm} \end{bmatrix} \begin{bmatrix} \mathbf{q}_p^{i1m} \\ \bar{\mathbf{q}}_p^{i1m} \end{bmatrix} \tag{31}$$

where $\mathbf{S}^{ilm} = \mathbf{I} \sin \hat{\psi}^{ilm}$, $\mathbf{C}^{ilm} = \mathbf{I} \cos \hat{\psi}^{ilm}$ and $\mathbf{I}$ is the 3×3 identity matrix. It remains to show that this candidate mode pair satisfies (8)–(10).

Substitution of (27) into (8) yields a equations that simplify to

$$\begin{bmatrix} (k_{cb,xx}^i - \omega^2 m_c^i) x_c^i \\ (k_{cb,yy}^i - \omega^2 m_c^i) y_c^i \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} x_c^i + \sum_{f=1, f\neq i}^{a} k_{c,xx}^{if}(x_c^i - x_c^f) + \sum_{j=1}^{b} k_{cg,xx}^{ij}(x_c^i - x_g^j) \\ \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} y_c^i + \sum_{f=1, f\neq i}^{a} k_{c,yy}^{if}(y_c^i - y_c^f) + \sum_{j=1}^{b} k_{cg,yy}^{ij}(y_c^i - y_g^j) \\ \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} (-r_c^{ilm} x_c^i \sin \psi^{ilm} + r_c^{ilm} y_c^i \cos \psi^{ilm}) \end{bmatrix} + \sum_{l=1}^{c^i} \mathbf{K}_{c,pt}^{il} \mathbf{q}_{pt}^{il} = 0, \quad i = 1, 2, \ldots, a \tag{32}$$

The last term in (32), which represents the planet trains, is expanded using (31) and the matrix definitions in Appendix B as

$$\begin{aligned} \sum_{l=1}^{c^i} \mathbf{K}_{c,pt}^{il} \mathbf{q}_{pt}^{il} &= \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \mathbf{K}_{cp}^{ilm} \mathbf{q}_p^{ilm} = \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \mathbf{K}_{cp}^{ilm} (\mathbf{q}_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\mathbf{q}}_p^{i1m} \sin \hat{\psi}^{ilm}) \\ &= \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} \begin{bmatrix} -(\zeta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\zeta}_p^{i1m} \sin \hat{\psi}^{ilm}) \cos \psi^{ilm} + (\eta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\eta}_p^{i1m} \sin \hat{\psi}^{ilm}) \sin \psi^{ilm} \\ -(\zeta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\zeta}_p^{i1m} \sin \hat{\psi}^{ilm}) \sin \psi^{ilm} - (\eta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\eta}_p^{i1m} \sin \hat{\psi}^{ilm}) \cos \psi^{ilm} \\ -r_c^{ilm} (\eta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\eta}_p^{i1m} \sin \hat{\psi}^{ilm}) \end{bmatrix} \end{aligned} \tag{33}$$

Insertion of (33) into (32), simplification using (12), and deletion of equations that are satisfied identically (the third equation) yields the two equations

$$\begin{aligned} &\begin{bmatrix} (k_{cb,xx}^i - \omega^2 m_c^i) x_c^i \\ (k_{cb,yy}^i - \omega^2 m_c^i) y_c^i \end{bmatrix} + \begin{bmatrix} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} x_c^i + \sum_{f=1, f\neq i}^{a} k_{c,xx}^{if}(x_c^i - x_c^f) + \sum_{j=1}^{b} k_{cg,xx}^{ij}(x_c^i - x_g^j) \\ \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} y_c^i + \sum_{f=1, f\neq i}^{a} k_{c,yy}^{if}(y_c^i - y_c^f) + \sum_{j=1}^{b} k_{cg,yy}^{ij}(y_c^i - y_g^j) \end{bmatrix} \\ &+ \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{i1m} \begin{bmatrix} -(\zeta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\zeta}_p^{i1m} \sin \hat{\psi}^{ilm}) \cos \psi^{ilm} + (\eta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\eta}_p^{i1m} \sin \hat{\psi}^{ilm}) \sin \psi^{ilm} \\ -(\zeta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\zeta}_p^{i1m} \sin \hat{\psi}^{ilm}) \sin \psi^{ilm} - (\eta_p^{i1m} \cos \hat{\psi}^{ilm} + \bar{\eta}_p^{i1m} \sin \hat{\psi}^{ilm}) \cos \psi^{ilm} \end{bmatrix} = 0 \quad i = 1, 2, \ldots, a \end{aligned} \tag{34}$$

Substitution of $\bar{\mathbf{q}}$ from (28) into (8) and use of the same simplification process as for $\mathbf{q}$ yields

$$\begin{aligned} &\begin{bmatrix} (k_{cb,xx}^i - \omega^2 m_c^i) y_c^i \\ -(k_{cb,yy}^i - \omega^2 m_c^i) x_c^i \end{bmatrix} + \begin{bmatrix} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} y_c^i + \sum_{f=1, f\neq i}^{a} k_{c,xx}^{if}(y_c^i - y_c^f) + \sum_{j=1}^{b} k_{cg,xx}^{ij}(y_c^i - y_g^j) \\ -\sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} x_c^i + \sum_{f=1, f\neq i}^{a} k_{c,yy}^{if}(x_c^f - x_c^i) + \sum_{j=1}^{b} k_{cg,yy}^{ij}(x_g^j - x_c^i) \end{bmatrix} \\ &+ \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{i1m} \begin{bmatrix} -(-\zeta_p^{i1m} \sin \hat{\psi}^{ilm} + \bar{\zeta}_p^{i1m} \cos \hat{\psi}^{ilm}) \cos \psi^{ilm} + (-\eta_p^{i1m} \sin \hat{\psi}^{ilm} + \bar{\eta}_p^{i1m} \cos \hat{\psi}^{ilm}) \sin \psi^{ilm} \\ -(-\zeta_p^{i1m} \sin \hat{\psi}^{ilm} + \bar{\zeta}_p^{i1m} \cos \hat{\psi}^{ilm}) \sin \psi^{ilm} - (-\eta_p^{i1m} \sin \hat{\psi}^{ilm} + \bar{\eta}_p^{i1m} \cos \hat{\psi}^{ilm}) \cos \psi^{ilm} \end{bmatrix} = 0 \quad i = 1, 2, \ldots, a \end{aligned} \tag{35}$$

Using (14), (16), and assumption 4, it can be shown that only $2a$ of the $4a$ equations in (34) and (35) are linearly independent (e.g., the first equation in (34) and the second equation in (35) can be shown to be identical except for a factor of -1). Thus, (34) and (35) can be represented by (34) alone.

Insertion of the candidate mode $\mathbf{q}$ of (27) into (9) yields

[1]Note the sign differences between (30) and the corresponding equation in Lin and Parker's derivation [4]. This is a typographical error in [4] that is corrected in this paper.

$$
\begin{bmatrix}
\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}(x_g^j\sin^2\psi_g^{jilm}-y_g^j\cos\psi_g^{jilm}\sin\psi_g^{jilm})+\sum_{n=1,n\neq j}^{b}k_{g,xx}^{nj}(x_g^j-x_g^n)+\sum_{i=1}^{a}k_{cg,xx}^{ij}(x_g^j-x_c^i)\\
\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}(-x_g^j\cos\psi_g^{jilm}\sin\psi_g^{jilm}+y_g^j\cos^2\psi_g^{jilm})+\sum_{n=1,n\neq j}^{b}k_{g,yy}^{nj}(y_g^j-y_g^n)+\sum_{i=1}^{a}k_{cg,yy}^{ij}(y_g^j-y_c^i)\\
\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}(-r_g^jx_g^j\sin\psi_g^{jilm}+r_g^jy_g^j\cos\psi_g^{jilm})
\end{bmatrix}
+\begin{bmatrix}(k_{gb,xx}^j-\omega^2m_g^j)x_g^j\\(k_{gb,yy}^j-\omega^2m_g^j)y_g^j\\0\end{bmatrix}
$$

$$
+\sum_{i=1}^{a}\sum_{l=1}^{c^i}\mathbf{K}_{g,pt}^{jil}\mathbf{q}_{pt}^{il}=0,\quad j=1,2,\ldots,b \tag{36}
$$

As before, the last term is expanded using (31) and the matrix definitions in Appendix B as

$$
\sum_{i=1}^{a}\sum_{l=1}^{c^i}\mathbf{K}_{g,pt}^{jil}\mathbf{q}_{pt}^{il}=\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\mathbf{K}_{gp}^{jilm}\mathbf{q}_{p}^{ilm}=\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\mathbf{K}_{gp}^{jilm}(\mathbf{q}_p^{i1m}\cos\hat{\psi}^{ilm}+\overline{\mathbf{q}}_p^{i1m}\sin\hat{\psi}^{ilm})=\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}\left(\cos\hat{\psi}^{ilm}\begin{bmatrix}z^{jilm}\sin\psi_g^{jilm}\\-z^{jilm}\cos\psi_g^{jilm}\\-r_g^jz^{jilm}\end{bmatrix}\right.
$$

$$
\left.+\sin\hat{\psi}^{ilm}\begin{bmatrix}\overline{z}^{jilm}\sin\psi_g^{jilm}\\-\overline{z}^{jilm}\cos\psi_g^{jilm}\\-r_g^j\overline{z}^{jilm}\end{bmatrix}\right)\quad j=1,2,\ldots,b \tag{37}
$$

$$
z^{jilm}=-\sigma^j\zeta_p^{i1m}\sin\alpha_g^{jilm}+\eta_p^{i1m}\cos\alpha_g^{jilm}+\sigma^ju_p^{i1m}
$$

$$
\overline{z}^{jilm}=-\sigma^j\overline{\zeta}_p^{i1m}\sin\alpha_g^{jilm}+\overline{\eta}_p^{i1m}\cos\alpha_g^{jilm}+\sigma^j\overline{u}_p^{i1m} \tag{38}
$$

where z^{jilm} and $\overline{z}^{jilm}$ are introduced for convenience. By Assumption 1, $z^{jilm}=z^{ji1m}$ for all i,j,l,m. Using this and (12), (37) simplifies to

$$
\sum_{i=1}^{a}\sum_{l=1}^{c^i}\mathbf{K}_{g,pt}^{jil}\mathbf{q}_{pt}^{jl}=\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}\begin{bmatrix}(z^{ji1m}\cos\hat{\psi}^{ilm}+\overline{z}^{ji1m}\sin\hat{\psi}^{ilm})\sin\psi_g^{jilm}\\-(z^{ji1m}\cos\hat{\psi}^{ilm}+\overline{z}^{ji1m}\sin\hat{\psi}^{ilm})\cos\psi_g^{jilm}\\0\end{bmatrix}\quad j=1,2,\ldots,b \tag{39}
$$

By substitution of (39) into (36) and use of (17) and (18) the third equation in (36) vanishes identically and the first two yield

$$
\begin{bmatrix}(k_{gb,xx}^j-\omega^2m_g^j)x_g^j\\(k_{gb,yy}^j-\omega^2m_g^j)y_g^j\end{bmatrix}+\begin{bmatrix}\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}x_g^i\sin^2\psi_g^{jilm}+\sum_{n=1,n\neq j}^{b}k_{g,xx}^{nj}(x_g^j-x_g^n)+\sum_{i=1}^{a}k_{cg,xx}^{ij}(x_g^j-x_c^i)\\\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}y_g^i\cos^2\psi_g^{jilm}+\sum_{n=1,n\neq j}^{b}k_{g,yy}^{nj}(y_g^j-y_g^n)+\sum_{i=1}^{a}k_{cg,yy}^{ij}(y_g^j-y_c^i)\end{bmatrix}
$$

$$
+\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}\begin{bmatrix}(z^{ji1m}\cos\hat{\psi}^{ilm}+\overline{z}^{ji1m}\sin\hat{\psi}^{ilm})\sin\psi_g^{jilm}\\-(z^{ji1m}\cos\hat{\psi}^{ilm}+\overline{z}^{ji1m}\sin\hat{\psi}^{ilm})\cos\psi_g^{jilm}\end{bmatrix}=0\quad j=1,2,\ldots,b \tag{40}
$$

Insertion of the candidate mode $\overline{\mathbf{q}}$ of (28) into (9) and use of the same simplification process as for $\mathbf{q}$ yields

$$
\begin{bmatrix}(k_{gb,xx}^j-\omega^2m_g^j)y_g^j\\-(k_{gb,yy}^j-\omega^2m_g^j)x_g^j\end{bmatrix}+\begin{bmatrix}\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}y_g^i\sin^2\psi_g^{jilm}+\sum_{n=1,n\neq j}^{b}k_{g,xx}^{nj}(y_g^j-y_g^n)+\sum_{i=1}^{a}k_{cg,xx}^{ij}(y_g^j-y_c^i)\\-\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}x_g^i\cos^2\psi_g^{jilm}+\sum_{n=1,n\neq j}^{b}k_{g,yy}^{nj}(x_g^n-x_g^j)+\sum_{i=1}^{a}k_{cg,yy}^{ij}(x_c^i-x_g^j)\end{bmatrix}
$$

$$
+\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\kappa_{gp}^{jilm}\begin{bmatrix}(-z^{ji1m}\sin\hat{\psi}^{ilm}+\overline{z}^{ji1m}\cos\hat{\psi}^{ilm})\sin\psi_g^{jilm}\\-(-z^{ji1m}\sin\hat{\psi}^{ilm}+\overline{z}^{ji1m}\cos\hat{\psi}^{ilm})\cos\psi_g^{jilm}\end{bmatrix}=0\quad j=1,2,\ldots,b \tag{41}
$$

By use of (19), (20), (21), and assumption 4, only $2b$ equations of the $4b$ equations in (40) and (41) are independent. Satisfaction of (40) implies satisfaction of (41), so (40) and (41) and can be represented by (40) alone.

Finally (27) and (28) are substituted into (10) and expanded using the matrix definitions in Appendix B. For notational convenience, two new variables $\mathbf{L}^{ilm}$ and $\overline{\mathbf{L}}^{ilm}$ are introduced to describe the result

$$\mathbf{L}^{ilm} = \sum_{n=1}^{d^i} \mathbf{K}_p^{ilmn}\mathbf{q}_p^{iln} - \omega^2 \mathbf{M}_p^{ilmm}\mathbf{q}_p^{ilm} + (\mathbf{K}_{cp}^{ilm})^T\mathbf{q}_c^i + \sum_{j=1}^{b} (\mathbf{K}_{gp}^{jilm})^T\mathbf{q}_g^j = 0 \qquad i=1,2,\ldots,a, \quad l=1,2,\ldots c^i$$
$$\overline{\mathbf{L}}^{ilm} = \sum_{n=1}^{d^i} \mathbf{K}_p^{ilmn}\overline{\mathbf{q}}_p^{iln} - \omega^2 \mathbf{M}_p^{ilmm}\overline{\mathbf{q}}_p^{ilm} + (\mathbf{K}_{cp}^{ilm})^T\overline{\mathbf{q}}_c^i + \sum_{j=1}^{b} (\mathbf{K}_{gp}^{jilm})^T\overline{\mathbf{q}}_g^j = 0 \qquad m=1,2,\ldots,d^i \tag{42}$$

The following identity is observed

$$(\mathbf{K}_{cp}^{ilm})^T\mathbf{q}_c^i = \cos\hat{\psi}^{ilm}(\mathbf{K}_{cp}^{i1m})^T\mathbf{q}_c^i + \sin\hat{\psi}^{ilm}(\mathbf{K}_{cp}^{i1m})^T\overline{\mathbf{q}}_c^i \tag{43}$$

This is verified by expanding $\mathbf{K}_{cp}^{ilm}$ according to the definition in Appendix B, substituting $\hat{\psi}^{ilm}=\psi^{ilm}-\psi^{i1m}$, expanding $\sin(x+y)$ and $\cos(x+y)$, and simplifying using $\sin^2 x+\cos^2 x=1$. This and a similar identity for $(\mathbf{K}_{gp}^{ilm})^T$ along with (11) and (31) allows (42) to be written as

$$\mathbf{L}^{ilm} = \sum_{n=1}^{d^i} \mathbf{K}_p^{i1mn}(\mathbf{C}^{ilm}\mathbf{q}_p^{i1n} + \mathbf{S}^{ilm}\overline{\mathbf{q}}_p^{i1n}) - \omega^2\mathbf{M}_p^{i1mn}(\mathbf{C}^{ilm}\mathbf{q}_p^{i1m} + \mathbf{S}^{ilm}\overline{\mathbf{q}}_p^{i1m}) + (\mathbf{K}_{cp}^{i1m})^T(\mathbf{C}^{ilm}\mathbf{q}_c^i + \mathbf{S}^{ilm}\overline{\mathbf{q}}_c^i) + \sum_{j=1}^{b}(\mathbf{K}_{gp}^{ji1m})^T(\mathbf{C}^{ilm}\mathbf{q}_g^j + \mathbf{S}^{ilm}\overline{\mathbf{q}}_g^j) = 0$$

$$\overline{\mathbf{L}}^{ilm} = \sum_{n=1}^{d^i} \mathbf{K}_p^{i1mn}(-\mathbf{S}^{ilm}\mathbf{q}_{pt}^{i1n} + \mathbf{C}^{ilm}\overline{\mathbf{q}}_p^{i1n}) - \omega^2\mathbf{M}_p^{i1mn}(-\mathbf{S}^{ilm}\mathbf{q}_p^{i1m} + \mathbf{C}^{ilm}\overline{\mathbf{q}}_p^{i1m}) + (\mathbf{K}_{cp}^{i1m})^T(-\mathbf{S}^{ilm}\mathbf{q}_c^i + \mathbf{C}^{ilm}\overline{\mathbf{q}}_c^i) + \sum_{j=1}^{b}(\mathbf{K}_{gp}^{ji1m})^T(-\mathbf{S}^{ilm}\mathbf{q}_g^j + \mathbf{C}^{ilm}\overline{\mathbf{q}}_g^j) = 0$$

$$i=1,2,\ldots,a \qquad l=1,2,\ldots,c^i \qquad m=1,2,\ldots,d^i \tag{44}$$

Factoring out common terms yields

$$\mathbf{L}^{ilm} = \mathbf{C}^{ilm}\mathbf{L}^{i1m} + \mathbf{S}^{ilm}\overline{\mathbf{L}}^{i1m} = \cos\psi^{ilm}\mathbf{L}^{i1m} + \sin\psi^{ilm}\overline{\mathbf{L}}^{i1m} = 0 \qquad i=1,2,\ldots,a$$
$$\overline{\mathbf{L}}^{ilm} = -\mathbf{S}^{ilm}\mathbf{L}^{i1m} + \mathbf{C}^{ilm}\overline{\mathbf{L}}^{i1m} = -\sin\psi^{ilm}\mathbf{L}^{i1m} + \cos\psi^{ilm}\overline{\mathbf{L}}^{i1m} = 0 \qquad l=1,2,\ldots,c^i \quad m=1,2,\ldots,d^i \tag{45}$$

This shows that $\mathbf{L}^{ilm}$ and $\overline{\mathbf{L}}^{ilm}$ for $l>1$ can each be written as a linear combination of $\mathbf{L}^{i1m}$ and $\overline{\mathbf{L}}^{i1m}$, and (45) represents, in general, only $6\Sigma_{i=1}^{a}d^i$ independent equations from $\mathbf{L}^{i1m}=0$ and $\overline{\mathbf{L}}^{i1m}=0$. Thus, (34), (40), and (45) form a $2a+2b+6\Sigma_{i=1}^{a}d^i$ degree of freedom eigenvalue problem.

The eigenvector for this reduced order eigenvalue problem is

$$[x_c^1\ y_c^1\ x_c^2\ y_c^2 \cdots x_c^a\ x_c^a | x_g^1\ y_g^1 \cdots x_g^b\ y_g^b | \mathbf{q}_{pt}^{11}\ \mathbf{q}_{pt}^{21} \cdots \mathbf{q}_{pt}^{a1} | \overline{\mathbf{q}}_{pt}^{11} \cdots \overline{\mathbf{q}}_{pt}^{a1}]^T$$

where $\mathbf{q}_{pt}^{il}$ is defined in (7). From each eigenvector of this reduced problem, two eigenvectors of the full problem are generated according to (30) and (31). If $\mathbf{q}$ and $\overline{\mathbf{q}}$ in (27)–(31) are interchanged, the eigenvectors of the full system (3) must remain the same (i.e., the choice of which vector is $\mathbf{q}$ and which is $\overline{\mathbf{q}}$ is arbitrary). Thus, each eigenvalue of the reduced problem has multiplicity two (except for special parameter combinations where two or more degenerate eigenvalue pairs happen to coincide). For each degenerate eigenvalue of the reduced problem, both reduced problem eigenvectors generate identical eigenvectors of the full problem.

Thus it has been shown that if all planet sets have three or more planet trains, then there are $a+b+3\Sigma_{i=1}^{a}d^i$ numerically different translational natural frequencies, each with multiplicity two.

3.3 Planet Modes. By describing each planet train's motion as a scalar multiple of the arbitrarily chosen first planet train's motion, a candidate planet mode associated with planet set i is written in the form

$$\mathbf{q}^i = [\mathbf{0}\cdots\mathbf{0}|\mathbf{0}\cdots\mathbf{0}|\mathbf{0}\cdots\mathbf{q}_{ps}^i\ \mathbf{0}\cdots\mathbf{0}]^T \tag{46}$$

$$\mathbf{q}_{ps}^i = [w^1\mathbf{q}_{pt}^{i1}\ w^2\mathbf{q}_{pt}^{i1}\ w^3\mathbf{q}_{pt}^{i1}\cdots w^{c^i}\mathbf{q}_{pt}^{i1}]^T \tag{47}$$

with $\mathbf{q}_{pt}^{i1}$ defined in (7) and $i\in\{1,2,\ldots a\}$ denoting a particular planet set. Insertion of (46) and (47) into (8)–(10), expansion using matrix definitions from Appendix B, and use of (38) leads to

$$\sum_{l=1}^{c^i}\mathbf{K}_{c,pt}^{il}w^l\mathbf{q}_{pt}^{i1} = \sum_{l=1}^{c^i}w^l\sum_{m=1}^{d^i}k_p^{i1m}\begin{bmatrix}-\zeta_p^{i1m}\cos\psi^{ilm}+\eta_p^{i1m}\sin\psi^{ilm}\\ -\zeta_p^{i1m}\sin\psi^{ilm}-\eta_p^{i1m}\cos\psi^{ilm}\\ -r_c^{i1m}\eta_p^{i1m}\end{bmatrix}=0 \tag{48}$$

$$\sum_{l=1}^{c^i}\mathbf{K}_{g,pt}^{jil}w^l\mathbf{q}_{pt}^{i1} = \sum_{l=1}^{c^i}w^l\sum_{m=1}^{d^i}\kappa_{gp}^{ji1m}z^{ji1m}\begin{bmatrix}\sin\psi_g^{jilm}\\ -\cos\psi_g^{jilm}\\ -r_g^j\end{bmatrix}=0$$

$$j=1,2,\ldots,b \tag{49}$$

$$(\mathbf{K}_{pt}^{il}-\omega^2\mathbf{M}_{pt}^{il})w^l\mathbf{q}_{pt}^{i1}=0, \quad l=1,2,\ldots,c^i \tag{50}$$

For non-trivial solutions (47), $w^l\neq 0$ for at least one $l\in\{1,2,\ldots,c^i\}$. Using this and (11), (50) simplifies to the same eigenvalue problem for any such l. Namely

$$(\mathbf{K}_{pt}^{i1}-\omega^2\mathbf{M}_{pt}^{i1})\mathbf{q}_{pt}^{i1}=0 \tag{51}$$

This is the reduced order eigenvalue problem for the motion of a single planet train. There are $3d^i$ eigensolutions of the reduced problem (51). The eigenvalues are distinct, in general, but they could be degenerate for special parameter combinations.

It remains to determine the w^l in (47) and satisfy (48) and (49). The first two equations in (48) represent forces that the planets exert on the carrier for deflection in a given mode. At each planet, the forces are represented in a local coordinate system (i.e., radial and tangential directions), resolved into the global x and y directions, and then summed. This can be simplified by resolving each force into an intermediate coordinate system. The resultant forces on the carrier in the x and y directions from the lth planet train are

$$F_x^{il} = w^l \sum_{m=1}^{d} k_p^{i1m}(-\zeta_p^{i1m}\cos\psi^{jlm} + \eta_p^{i1m}\sin\psi^{jlm})$$

$$F_y^{il} = w^l \sum_{m=1}^{d} k_p^{i1m}(-\zeta_p^{i1m}\sin\psi^{jlm} - \eta_p^{i1m}\cos\psi^{jlm}) \tag{52}$$

These forces resolved in the radial and tangential directions of the (arbitrarily chosen) first planet are

$$F_\zeta^{il} = F_x^{il}\cos\psi^{jl1} + F_y^{il}\sin\psi^{jl1}$$

$$F_\eta^{il} = -F_x^{il}\sin\psi^{jl1} + F_y^{il}\cos\psi^{jl1} \tag{53}$$

Substitution of (52) into (53) and simplification using trigonometric identities yields

$$F_\zeta^{il} = w^l \sum_{m=1}^{d^i} [-\zeta_p^{i1m}k_p^{i1m}\cos(\psi^{jlm}-\psi^{jl1}) + \eta_p^{i1m}k_p^{i1m}\sin(\psi^{jlm}-\psi^{jl1})]$$

$$F_\eta^{il} = w^l \sum_{m=1}^{d^i} [-\zeta_p^{i1m}k_p^{i1m}\sin(\psi^{jlm}-\psi^{jl1}) - \eta_p^{i1m}k_p^{i1m}\cos(\psi^{jlm}-\psi^{jl1})] \tag{54}$$

The sums in (54) are the same for every l, thus $F_\zeta^{il}=w^l F_\zeta^{i1}$ and $F_\eta^{il}=w^l F_\eta^{i1}$ (where $w^1=1$ without loss of generality). Then, resolving these forces back into the x and y directions (i.e., inverting (53)), the first two equations in (48) are expressed as

$$\begin{bmatrix} F_\zeta^{i1} & -F_\eta^{i1} \\ F_\eta^{i1} & F_\zeta^{i1} \end{bmatrix} \begin{bmatrix} \sum_{l=1}^{c^i} w^l\cos\psi^{jl1} \\ \sum_{l=1}^{c^i} w^l\sin\psi^{jl1} \end{bmatrix} = 0 \tag{55}$$

The determinant $(F_\eta^{i1})^2+(F_\zeta^{i1})^2$ of the matrix in (55) is nonzero unless the planet modal deflections ζ_p^{i1m} and η_p^{i1m} are such that each planet train exerts zero resultant force on the carrier (i.e., $F_\eta^{i1}=F_\zeta^{i1}=0$). These modal deflections are determined by (51), however, which involves the planet inertias, bearing stiffnesses, and mesh stiffnesses, so the deflections will not, in general, satisfy $F_\eta^{i1}=F_\zeta^{i1}=0$, which are independent of (51) and do not involve inertias or mesh stiffnesses. Thus the matrix is, in general, invertible and the two sums in (55) each vanish.

The third equation in (48) represents the moment exerted on the carrier by the planets. Satisfaction of this equation requires either $\Sigma_{l=1}^{c^i} w^l=0$ or $\Sigma_{m=1}^{d^i} k_p^{i1m} r_c^{i1m}\eta_p^{i1m}=0$. The second of these is independent of the $3d^i$ equations of (51) that determine the η_p^{i1m} and so will not be satisfied, in general. Thus, $\Sigma_{l=1}^{c^i} w^l=0$. Therefore, in the general case, three constraints on the w^l have been obtained.

$$\sum_{l=1}^{c^i} w^l\sin\psi^{jl1}=0 \quad \sum_{l=1}^{c^i} w^l\cos\psi^{jl1}=0 \quad \sum_{l=1}^{c^i} w^l=0 \tag{56}$$

A similar procedure shows that (49) gives the same three equations.

Ambarisha and Parker [9] proved that for a simple planetary gear the number of independent solutions of (56) is exactly c^i-3. Thus, planet modes exist only for planet sets with four or more planet trains. Wu and Parker [7] give the following c^i-3 closed-form, independent solutions based on their consideration of simple planetary gears.

$$w^l = \cos\frac{(n+1)2\pi(l-1)}{c^i}, \quad n=1,2,\ldots,\left\lceil\frac{c^i-3}{2}\right\rceil \tag{57}$$

$$w^l = \sin\frac{(n+1)2\pi(l-1)}{c^i}, \quad n=1,2,\ldots,\left\lfloor\frac{c^i-3}{2}\right\rfloor \tag{58}$$

where the brackets in (57) and (58) denote the ceiling and floor functions, respectively.

For each eigensolution of the reduced problem (51), c^i-3 independent vibration modes of the full problem (3), each having the same natural frequency, are constructed from (46) and (47), and the c^i-3 independent solutions of (56). Thus, for each planet set $i=1,2,\ldots,a$ there are $3(c^i-3)d^i$ vibration modes of the form (46) and (47); these are grouped into $3d^i$ sets of degenerate modes, where each set has an associated natural frequency of multiplicity c^i-3.

For special parameter combinations, it may happen that a rotational mode or translational mode may have no motion of the carrier(s) and central gears. These modes, however, should not be considered as planet modes because the planet motions satisfy (22) or (31) and not (47), (57), and (58). For example, in the system of Fig. 6 and Table 1, if the pressure angles of all of the gear-planet meshes are changed to zero (i.e., $\alpha_g^{jilm}=0$ for all i,j,l,m), then there is a pair of modes with frequency 5811.5 Hz that have no motion of the carriers or central gears. One mode in this pair is a planet mode with planet motions governed by (47), (57), and (58), but the other should be considered a rotational mode because its planet motions are governed by (22).[2]

3.4 Final Calculations. The total number of modes in the three mode types considered is $3(a+b+\Sigma_{i=1}^{a} c^i d^i)$. This equals the total number of degrees of freedom of the general compound planetary gear system. Therefore, the above set of modes is complete, and all modes can be classified as one of a rotational, translational, or planet mode.

3.5 Diametrically Opposed Planet Positions. In certain situations, assembly constraints cause the planet trains to be unequally spaced around the carrier. In the most general case, this causes the rotational and translational modes to couple and lose their distinctive properties (because (12)–(21) no longer hold). One special case of interest is the diametrically opposed case in which pairs of planets trains are 180 deg apart. In other words, for some i, $c^i\geqslant 2$, c^i is even, and $\psi^{j(l+c^i/2)m}=\pi+\psi^{jlm}$. This is common in practice. Each pair may be arbitrarily spaced. $c^i=2$ is allowed, which is an exception to assumption 3. For $c^i=2$, the terms "equally spaced" and "diametrically opposed" are equivalent. The properties of such a planet set follow the results for the diametrically opposed case.

For this spacing, the vibration modes retain a structure that is similar to the equally spaced case. Lin and Parker considered diametrically opposed planets for a simple, single stage planetary gears in [12]. Those results are now generalized to the compound case.

Assume, without loss of generality, that planet set 1 has diametrically opposed planet trains. Any other planet sets may be either equally spaced or diametrically opposed. Consider the effect of this spacing on the simplifications involving planet position angles (12)–(21). Equations (12) and (17) still hold for all i but the others do not. The proofs of the properties of the rotational modes and planet modes do not involve (13)–(16) or (18)–(21), thus, the properties of these modes retain their properties in the diametrically opposed case. The translational modes, however, do depend on these equations and thus their properties are modified for the diametrically opposed case. Specifically, translational modes have distinct eigenvalues instead of multiplicity two.

[2]Some numerical solvers may return coupled eigenvectors for this example degenerate pair. The results will not look like either planet modes or rotational modes. They can be decoupled by an orthogonalization process.

The candidate carrier and gear deflections for a translational mode are

$$\mathbf{q}_c^i = [x_c^i \; y_c^i \; 0]^T, \quad i = 1,2,\ldots,a \tag{59}$$

$$\mathbf{q}_g^j = [x_g^j \; y_g^j \; 0]^T, \quad j = 1,2,\ldots,b \tag{60}$$

The candidate planet motions are linear combinations of the planet motions of the (arbitrarily chosen) first two planet trains, that is

$$\mathbf{q}_p^{ilm} = \begin{bmatrix} \zeta_p^{ilm} \\ \eta_p^{ilm} \\ u_p^{ilm} \end{bmatrix} = f^{il}\begin{bmatrix} \zeta_p^{i1m} \\ \eta_p^{i1m} \\ u_p^{i1m} \end{bmatrix} + g^{il}\begin{bmatrix} \zeta_p^{i2m} \\ \eta_p^{i2m} \\ u_p^{i2m} \end{bmatrix} \quad \text{for all } i,l,m$$

where f^{il} and g^{il} are independent of $m \in \{1,2,\ldots d^i\}$. In terms of planet trains this is

$$\mathbf{q}_{pt}^{il} = f^{il}\mathbf{q}_{pt}^{i1} + g^{il}\mathbf{q}_{pt}^{i2} \quad \text{for all } i,l \tag{61}$$

For planet sets with three or more planet trains, f^{il} and g^{il} are defined as

$$\left.\begin{aligned} f^{il} &= -\frac{\sin(\psi^{ilm}-\psi^{i2m})}{\sin(\psi^{i2m}-\psi^{i1m})} \\ g^{il} &= \frac{\sin(\psi^{ilm}-\psi^{i1m})}{\sin(\psi^{i2m}-\psi^{i1m})} \end{aligned}\right. \quad \text{for all } i \text{ such that } c^i \geq 3 \tag{62}$$

For planet sets with two planet trains[3]

$$f^{i1} = 1, \quad f^{i2} = -1, \quad g^{i1} = g^{i2} = 0 \quad \text{for all } i \text{ such that } c^i = 2 \tag{63}$$

In both cases the following relations hold:

$$\sum_{l=1}^{c^i} f^{il} = \sum_{l=1}^{c^i} g^{il} = 0 \quad \text{for all } i \tag{64}$$

Start by substituting (59)–(61) into (8) and simplifying using Assumption 1, which yields

$$\begin{bmatrix} (k_{cb,xx}^i - \omega^2 m_c^i)x_c^i \\ (k_{cb,yy}^i - \omega^2 m_c^i)y_c^i \\ 0 \end{bmatrix} + \begin{bmatrix} \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} k_p^{i1m}x_c^i + \sum_{f=1,f\neq i}^{a} k_{c,xx}^{if}(x_c^i - x_c^f) + \sum_{j=1}^{b} k_{cg,xx}^{ij}(x_c^i - x_g^j) \\ \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} k_p^{i1m}y_c^i + \sum_{f=1,f\neq i}^{a} k_{c,yy}^{if}(y_c^i - y_c^f) + \sum_{j=1}^{b} k_{cg,yy}^{ij}(y_c^i - y_g^j) \\ \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} (-r_c^{i1m}x_c^i \sin\psi^{ilm} + r_c^{i1m}y_c^i\cos\psi^{ilm}) \end{bmatrix}$$
$$+ \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} k_p^{i1m}\begin{bmatrix} -(f^{il}\zeta_p^{i1m} + g^{il}\zeta_p^{i2m})\cos\psi^{ilm} + (f^{il}\eta_p^{i1m} + g^{il}\eta_p^{i2m})\sin\psi^{ilm} \\ -(f^{il}\zeta_p^{i1m} + g^{il}\zeta_p^{i2m})\sin\psi^{ilm} - (f^{il}\eta_p^{i1m} + g^{il}\eta_p^{i2m})\cos\psi^{ilm} \\ -r_c^{i1m}(f^{il}\eta_p^{i1m} + g^{il}\eta_p^{i2m}) \end{bmatrix} = 0, \quad i = 1,2,\ldots,a \tag{65}$$

Simplification using (12) and (64) and deletion of equations that are satisfied identically (the third equation) yields

$$\begin{bmatrix} (k_{cb,xx}^i - \omega^2 m_c^i)x_c^i \\ (k_{cb,yy}^i - \omega^2 m_c^i)y_c^i \end{bmatrix} + \begin{bmatrix} \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} k_p^{i1m}x_c^i + \sum_{f=1,f\neq i}^{a} k_{c,xx}^{if}(x_c^i - x_c^f) + \sum_{j=1}^{b} k_{cg,xx}^{ij}(x_c^i - x_g^j) \\ \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} k_p^{i1m}y_c^i + \sum_{f=1,f\neq i}^{a} k_{c,xx}^{if}(y_c^i - y_c^f) + \sum_{j=1}^{b} k_{cg,yy}^{ij}(y_c^i - y_g^j) \end{bmatrix}$$
$$+ \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} k_p^{i1m}\begin{bmatrix} -(f^{il}\zeta_p^{i1m} + g^{il}\zeta_p^{i2m})\cos\psi^{ilm} + (f^{il}\eta_p^{i1m} + g^{il}\eta_p^{i2m})\sin\psi^{ilm} \\ -(f^{il}\zeta_p^{i1m} + g^{il}\zeta_p^{i2m})\sin\psi^{ilm} - (f^{il}\eta_p^{i1m} + g^{il}\eta_p^{i2m})\cos\psi^{ilm} \end{bmatrix} = 0, \quad i = 1,2,\ldots,a \tag{66}$$

Insertion of (59)–(61) into (9) and a similar reduction yields

$$\begin{bmatrix} (k_{gb,xx}^j - \omega^2 m_g^j)x_g^j \\ (k_{gb,yy}^j - \omega^2 m_g^j)y_g^j \end{bmatrix} + \begin{bmatrix} \sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i} \kappa_{gp}^{ji1m}x_g^j\sin^2\psi_g^{jilm} + \sum_{n=1,n\neq j}^{b} k_{g,xx}^{jn}(x_g^j - x_g^n) + \sum_{i=1}^{a} k_{cg,xx}^{ij}(x_g^j - x_c^i) \\ \sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i} \kappa_{gp}^{ji1m}y_g^j\cos^2\psi_g^{jilm} + \sum_{n=1,n\neq j}^{b} k_{g,xx}^{jn}(y_g^j - y_g^n) + \sum_{i=1}^{a} k_{cg,yy}^{ij}(y_g^j - y_c^i) \end{bmatrix} + \sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i} \kappa_{gp}^{ji1m}\begin{bmatrix} \tilde{z}^{jilm}\sin\psi_g^{jilm} \\ -\tilde{z}^{jilm}\cos\psi_g^{jilm} \end{bmatrix}$$
$$= 0, \quad j = 1,2,\ldots,b$$

$$\tilde{z}^{jilm} = -\sigma^j(f^{il}\zeta_p^{i1m} + g^{il}\zeta_p^{i2m})\sin\alpha_g^{jilm} + (f^{il}\eta_p^{i1m} + g^{il}\eta_p^{i2m})\cos\alpha_g^{jilm} + \sigma^j(f^{il}u_p^{i1m} + g^{il}u_p^{i2m}) \tag{67}$$

[3]Equation (62) is similar to the result given in [12], but (63) is not given in [12]. This is an omission in [12] that is corrected here.

where the definition of $\tilde{z}^{jilm}$ is introduced for convenience.

Finally, substitution of (59)–(61) into (10) gives

$$\mathbf{N}^{il} = \{\mathbf{K}_{pt}^{il} - \omega^2 \mathbf{M}_{pt}^{il}\}(f^{il}\mathbf{q}_{pt}^{i1} + g^{il}\mathbf{q}_{pt}^{i2}) + (\mathbf{K}_{c,pt}^{il})^T \begin{bmatrix} x_c^i \\ y_c^i \\ 0 \end{bmatrix} + \sum_{j=1}^{b} (\mathbf{K}_{g,pt}^{jil})^T \begin{bmatrix} x_g^j \\ y_g^j \\ 0 \end{bmatrix} = 0, \quad \begin{matrix} i = 1,2, \ldots, a \\ l = 1,2, \ldots, c^i \end{matrix} \tag{68}$$

where $\mathbf{N}^{il}$ is introduced for convenience.

For all planet sets i the following identity holds:

$$(\mathbf{K}_{c,pt}^{il})^T \begin{bmatrix} x_c^i \\ y_c^i \\ 0 \end{bmatrix} = f^{il}(\mathbf{K}_{c,pt}^{i1})^T \begin{bmatrix} x_c^i \\ y_c^i \\ 0 \end{bmatrix} + g^{il}(\mathbf{K}_{c,pt}^{i2})^T \begin{bmatrix} x_c^i \\ y_c^i \\ 0 \end{bmatrix} \tag{69}$$

This is confirmed by expanding the matrix components on each side according to the definitions in Appendix B to yield

$$-x_c^i \cos \psi^{ilm} - y_c^i \sin \psi^{ilm} = -x_c^i(f^{il} \cos \psi^{i1m} + g^{il} \cos \psi^{i2m}) - y_c^i(g^{il} \sin \psi^{i2m} + f^{il} \sin \psi^{i1m})$$

$$x_c^i \sin \psi^{ilm} - y_c^i \cos \psi^{ilm} = x_c^i(f^{il} \sin \psi^{i1m} + g^{il} \sin \psi^{i2m}) - y_c^i(g^{il} \cos \psi^{i2m} + f^{il} \cos \psi^{i1m})$$

These equations hold for any values of x_c^i and y_c^i if and only if

$$\cos \psi^{ilm} = f^{il} \cos \psi^{i1m} + g^{il} \cos \psi^{i2m}$$

$$\sin \psi^{ilm} = f^{il} \sin \psi^{i1m} + g^{il} \sin \psi^{i2m} \tag{70}$$

Substituting the definitions of f^{il} and g^{il} in (62) (or (63), as appropriate) into (70) shows that it, and consequently (69), is identically satisfied. An identity similar to (69) can be shown for $\mathbf{K}_{g,pt}^{il}$.

These two identities along with (11) allow (68) to be written as

$$\mathbf{N}^{il} = \{\mathbf{K}_{pt}^{i1} - \omega^2 \mathbf{M}_{pt}^{i1}\}(f^{il}\mathbf{q}_{pt}^{i1} + g^{il}\mathbf{q}_{pt}^{i2}) + (f^{il}\mathbf{K}_{c,pt}^{i1} + g^{il}\mathbf{K}_{c,pt}^{i2})^T \begin{bmatrix} x_c^i \\ y_c^i \\ 0 \end{bmatrix} + \sum_{j=1}^{b} (f^{il}\mathbf{K}_{g,pt}^{ji1} + g^{il}\mathbf{K}_{g,pt}^{ji2})^T \begin{bmatrix} x_g^i \\ y_g^i \\ 0 \end{bmatrix} = f^{il}\mathbf{N}^{i1} + g^{il}\mathbf{N}^{i2}$$

$$i = 1,2, \ldots, a$$
$$l = 1,2, \ldots, c^i$$

Because $\mathbf{N}^{il}$ for any l can be written in terms of $\mathbf{N}^{i1}$ and $\mathbf{N}^{i2}$, (68) represents exactly $6d^i$ linearly independent equations for all planet sets such that $c^i \geqslant 3$, and $3d^i$ linearly independent equations for all planet sets such that $c^i = 2$.

Consequently, (66)–(68) form $2a+2b+6\Sigma_{i,c^i \geqslant 3} d^i + 3\Sigma_{i,c^i=2} d^i$ linear, homogenous equations for the same number of unknowns with parameter ω^2. This is the reduced order eigenvalue problem for the translational modes when one or more planet sets have only two planet trains. Thus it has been shown that if any one or more planet sets have diametrically opposed planet trains, even if all other planet sets have equally spaced planet trains, then there are exactly $2a+2b+6\Sigma_{i,c^i \geqslant 3} d^i + 3\Sigma_{i,c^i=2} d^i$ translational modes of the form (59)–(61), each with an associated natural frequency that is, in general, distinct.

3.6 Sensitivity to Relative Planet Set Orientation. Because the above analysis places no restrictions on relative angular orientations between the planet sets of multi-stage gears, the derived modal properties are the same for any relative orientations. The question remains, however, whether or not the natural frequencies and vibration modes themselves, not just their properties, are affected by the relative planet set orientations. In the most general multi-stage case, the eigensolutions vary with relative orientation, but they do not for systems satisfiying assumptions 1–4. Each vibration mode type is considered separately.

For rotational modes, expanding the reduced order eigenvalue problem of (23), (24), and (26) using matrix definitions in Appendix B shows that it does not explicitly or implicitly involve the planet position angles. Therefore, rotational natural frequencies and vibration modes do not change with relative planet set orientations.

Planet modes are derived from the reduced order eigenvalue problem (51) and the w^l satisfying (57) and (58). The reduced problem eigenvectors $\mathbf{q}_{pt}^{i1}$ and the natural frequencies ω do not depend on the planet positions because $\mathbf{M}_{pt}^{il}$ and $\mathbf{K}_{pt}^{il}$ in (51) are independent of the planet positions. Equations (57) and (58) show that the w^l do not depend on planet positions. Thus, the planet natural frequencies and vibration modes are independent of relative planet set orientation. This is expected as the deflections are confined to individual planet sets.

The translational modes are more difficult to consider analytically. The reduced order eigenvalue problems from (34), (40), and (45) (for equally spaced planet trains) and, (66)–(68) (for diametrically opposed planet trains) involve the planet position angles explicitly in summations of functions of the planet positions that change for different planet positions. An eigensensitivity analysis (as performed by Lin and Parker in [15]) would show the sensitivities of the natural frequencies to planet positions. Such an analysis is beyond the scope of this paper. On physical grounds, however, it can be seen that the degenerate modes in the equally spaced case are the result of the translational stiffness and inertia being isotropic (i.e., independent of the direction of translational motion). After rotation of any carrier, this isotropic relationship still holds. Thus, the natural frequencies and mode shapes are independent of relative planet set orientation. For the diametrically opposed case, however, the translational modes are not degenerate. Thus, a change in the relative orientation of the planet sets will change the natural frequencies and vibration modes. These results have been verified in numerical examples.

4 Discussion

While the mathematical proof is of interest mainly to the academic researcher, the results are useful for both the practical gearbox designer and the academic researcher. To a gearbox designer, the prediction of natural frequencies allows resonance conditions to be avoided when designing a planetary gear system. The classification of modes into various types and knowledge of the number of numerically different natural frequencies is also important for avoiding resonant response, reducing excitation of particular mode types, and understanding whether response in a particular mode will generate torque (rotational modes), force (translational modes), or neither (planet modes) to the structures supporting the central gears (sun and ring). In cases where design constraints require operating tooth mesh frequencies to be near natural frequencies, the use of *planet mesh phasing* can suppress resonant response [9,14,18,19]. Understanding and application of planet mesh phasing depends on the unique properties of planetary gear free vibration as shown here. Future work on understanding the various mode types may identify certain types of modes as having more significant impact on noise, fatigue life, and other factors of interest.

For research purposes, extension of the dynamic model of simple planetary gears to the general compound case allows other investigations of simple planetary gear vibration to be extended to the compound case. Research on nonlinear response, parametric excitation from fluctuating mesh stiffness, elastic ring deformation, etc. (e.g., [7,8,10,20–22]) can be carried out for general compound planetary gears. These analyses benefit from understanding the properties of the different mode types, particularly because such analyses frequently adopt the assumption that only one or two modes are present in the response (modal truncation). In that case, one can examine the interactions between two modes of the same or different types. For example, is a combination parametric instability possible between a rotational and translational mode?

5 Summary and Conclusions

A dynamic model of compound, multi-stage planetary gears of general description has been developed. The model has been used to examine the free vibration of compound, multi-stage planetary gears. For identical, equally spaced planet trains, the natural frequencies and vibration modes of compound planetary gears have highly structured properties due to the system's cyclic symmetry. Specifically, *all* vibration modes can be classified into one of three types: Rotational, translational, and planet modes. Rotational modes with distinct natural frequencies have pure rotation of the central gears and all planet trains in a given planet set move identically. Pairs of translational modes with degenerate natural frequencies have pure translation of the carriers and central gears, and motions of the planet trains in a pair of orthonormal vibration modes can be found by a simple transformation of the first planet train's motion. Planet modes have motion of the planets in one planet set only and no motion of any carriers or central gears; the multiplicity of their natural frequencies is dictated by the number of planet trains in a planet set. Reduced order eigenvalue problems for each mode type are given explicitly.

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Nomenclature

a = number of carriers = number of planet sets
b = number of central gears
c^i = number of planet trains in planet set i
d^i = number of planets per train in planet set i (number of planets must be the same in all trains)
e = static transmission error, L
$F(t)$ = externally applied force, F
I = mass moment of inertia, M-L^2
k_p^{ilm} = planet bearing stiffness of planet m in planet train l in planet set i, F/L
$k_{g,xx}^{jn}$, $k_{g,yy}^{jn}$ = shaft stiffness between gear j and gear n, F/L
$K_{g,\theta\theta}^{jn}$ = shaft stiffness between gear j and gear n, F-L/rad
$k_{c,xx}^{ih}$, $k_{c,yy}^{ih}$ = shaft stiffness between carrier i and carrier h, F/L
$K_{c,\theta\theta}^{ih}$ = shaft stiffness between carrier i and carrier h, F-L/rad
$k_{cg,xx}^{ij}$, $k_{cg,yy}^{ij}$ = shaft stiffness between carrier i and gear j, F/L
$K_{cg,\theta\theta}^{ij}$ = shaft stiffness between carrier i and gear j, F-L/rad
$k_{pp,\zeta\zeta}^{ilmn}$, $k_{pp,\eta\eta}^{ilmn}$ = shaft stiffness between planet m and n of planet set i planet train l, F/L
$K_{pp,uu}^{ilmn}$ = shaft stiffness between planet m and n of planet set i planet train l, F-L/rad
$k_{cb,xx}^{i}$, $k_{cb,yy}^{i}$ = bearing stiffness between carrier i and ground, F/L
$K_{cb,\theta\theta}^{i}$ = bearing stiffness between carrier i and ground, F-L/rad
$k_{gb,xx}^{j}$, $k_{gb,yy}^{j}$ = bearing stiffness between gear j and ground, F/L
$K_{gb,\theta\theta}^{j}$ = bearing stiffness between gear j and ground, F-L/rad
m = mass, M
r = radius (base radius for gears, radius to planet centers for carriers), L
u = rotational coordinate, $u=r\theta$, L
x, y = translational coordinate, L
α_g^{jilm} = pressure angle at the mesh between gear j and planet m of planet train l of planet set i, rad
α_p^{ilmn} = pressure angle at the mesh between planet m and planet n of planet train l of planet set i, rad
β^{ilmn} = for meshed planets: angle from the positive ζ direction of planet m to a line connecting planets m and n in planet train l, planet set i. Undefined for stepped planets. $\beta^{ilmm}=0$, rad
γ^{ilmn} = $\beta^{ilmn}-\alpha_p^{ilmn}$, rad
δ = deflection of an elastic element (compression positive), L
$\kappa_{gp}^{jilm}(t)$ = time varying mesh stiffness between gear j and planet m of planet train l of planet set i, F/L
$\kappa_{pp}^{ilmn}(t)$ = time varying mesh stiffness between planet m and planet n in planet train l of planet set i, F/L
σ^j = $\begin{cases} 1 \text{ if gear } j \text{ is a ring(internal gear)} \\ -1 \text{ if gear } j \text{ is a sun(external gear)} \end{cases}$
θ = rotational coordinate, rad
$\psi^{ilm}(t)$ = angular position of planet m of planet train l of planet set i, rad
$\hat{\psi}^{ilm}$ = $\psi^{ilm}-\psi^{i1m}$, rad
ψ_g^{jilm} = $\psi^{ilm}+\sigma^j\alpha_g^{jilm}$, rad
$\hat{\psi}_g^{jilm}$ = $\psi_g^{jilm}-\psi_g^{ji1m}$, rad
$\tau(t)$ = externally applied torque, F-L
ζ, η = planet radial and tangential coordinates, L

Subscripts

b = bearing (to ground)
c = carrier
g = gear
m = mesh
p = planet
ps = planet set
pt = planet train

Superscripts

i = carrier or planet set
j = central gear (sun or ring)
l = planet train
m = planet
n, f = depends on context

Appendix A: Equations of Motion

Carriers:

$$m_c^i\ddot{x}_c^i + k_{cb,xx}^i x_c^i + \sum_{l=1}^{c^i}\sum_{m=1}^{d^i} k_p^{ilm}[\delta_\zeta^{ilm}\cos\psi^{ilm}(t) - \delta_\eta^{ilm}\sin\psi^{ilm}(t)]$$
$$+\sum_{j=1}^{b} k_{cg,xx}^{ij}(x_c^i - x_g^j) + \sum_{n=1,n\neq i}^{a} k_{c,xx}^{in}(x_c^i - x_c^n) = F_{c,xx}^i(t)$$

$$m_c^i \ddot{y}_c^i + k_{cb,yy}^i y_c^i + \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} k_p^{ilm} [\delta_\zeta^{ilm} \sin \psi^{ilm}(t) + \delta_\eta^{ilm} \cos \psi^{ilm}(t)]$$

$$+ \sum_{j=1}^{b} k_{cg,yy}^{ij} (y_c^i - y_g^j) + \sum_{n=1,n\neq i}^{a} k_{c,yy}^{in} (y_c^i - y_c^n) = F_{c,yy}^i(t)$$

$$I_c^i \ddot{\theta}_c^i + K_{cb,\theta\theta}^i \theta_c^i + \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} r_c^{ilm} k_p^{ilm} \delta_\eta^{ilm} + \sum_{j=1}^{b} K_{cg,\theta\theta}^{ij} (\theta_c^i - \theta_g^j)$$

$$+ \sum_{n=1,n\neq i}^{a} K_{c,\theta\theta}^{in} (\theta_c^i - \theta_c^n) = \tau_c^i(t) \tag{A1}$$

Central gears (suns and rings). Same as (2), repeated here for completeness

$$m_g^j \ddot{x}_g^j + k_{gb,xx}^j x_g^j - \sum_{i=1}^{a} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} \sin \psi_g^{jilm}(t) + \sum_{i=1}^{a} k_{cg,xx}^{ij}$$

$$\times (x_g^j - x_c^i) + \sum_{n=1,n\neq j}^{b} k_{g,xx}^{jn} (x_g^j - x_g^n) = F_{g,xx}^j(t)$$

$$m_g^j \ddot{y}_g^j + k_{gb,yy}^j y_g^j + \sum_{i=1}^{a} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} \cos \psi_g^{jilm}(t) + \sum_{i=1}^{a} k_{cg,yy}^{ij}$$

$$\times (y_g^j - y_c^i) + \sum_{n=1,n\neq j}^{b} k_{g,yy}^{jn} (y_g^j - y_g^n) = F_{g,yy}^j(t)$$

$$I_g^j \ddot{\theta}_g^j + K_{gb,\theta\theta}^j \theta_g^j + \sum_{i=1}^{a} \sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} r_g^j + \sum_{i=1}^{a} K_{cg,\theta\theta}^{ij} (\theta_g^j - \theta_c^i)$$

$$+ \sum_{n=1,n\neq j}^{b} K_{g,\theta\theta}^{jn} (\theta_g^j - \theta_g^n) = \tau_g^j(t) \tag{A2}$$

Planets:

$$m_p^{ilm} \ddot{\zeta}_p^{ilm} - k_p^{ilm} \delta_\zeta^{ilm} - \sum_{j=1}^{b} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} \sin \alpha_g^{jilm}$$

$$+ \sum_{n=1}^{d^i} [-\kappa_{pp}^{ilmn}(t) \delta_p^{ilmn} \sin \gamma^{ilmn} + k_{pp,\zeta\zeta}^{ilmn} (\zeta_p^{ilm} - \zeta_p^{iln})] = 0$$

$$m_p^{ilm} \ddot{\eta}_p^{ilm} - k_p^{ilm} \delta_\eta^{ilm} + \sum_{j=1}^{b} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} \cos \alpha_g^{jilm}$$

$$+ \sum_{n=1}^{d^i} [\kappa_{pp}^{ilmn}(t) \delta_p^{ilmn} \cos \gamma^{ilmn} + k_{pp,\eta\eta}^{ilmn} (\eta_p^{ilm} - \eta_p^{iln})] = 0$$

$$\frac{I_p^{ilm}}{(r_p^{ilm})^2} \ddot{u}_p^{ilm} + \sum_{j=1}^{b} \kappa_{gp}^{jilm}(t) \delta_g^{jilm} + \sum_{n=1}^{d^i} \left[\kappa_{pp}^{ilmn}(t) \delta_p^{ilmn} + K_{pp,uu}^{ilmn} \right.$$

$$\left. \times \left(\frac{u_p^{ilm}}{(r_p^{ilm})^2} - \frac{u_p^{iln}}{r_p^{ilm} r_p^{iln}} \right) \right] = 0 \tag{A3}$$

Deflections of elastic elements (compression positive)

$$\delta_g^{jilm} = y_g^j \cos \psi_g^{jilm}(t) - x_g^j \sin \psi_g^{jilm}(t) + r_g^j \theta_g^j + \sigma^j \zeta_p^{ilm} \sin \alpha_g^{jilm}$$
$$- \eta_p^{ilm} \cos \alpha_g^{jilm} - \sigma^j u_p^{ilm} + e_{gp}^{jilm}(t) \tag{A4}$$

$$\delta_p^{ilmn} = -\zeta_p^{ilm} \sin \gamma^{ilmn} - \zeta_p^{iln} \sin \gamma^{ilnm} + \eta_p^{ilm} \cos \gamma^{ilmn} + \eta_p^{ijn} \cos \gamma^{ilnm}$$
$$+ u_p^{ilm} + u_p^{iln} + e_{pp}^{ilmn}(t) \tag{A5}$$

$$\delta_\zeta^{ilm} = x_c^i \cos \psi^{ilm}(t) + y_c^i \sin \psi^{ilm}(t) - \zeta_p^{ilm} \tag{A6}$$

$$\delta_\eta^{ilm} = -x_c^i \sin \psi^{ilm}(t) + y_c^i \cos \psi^{ilm}(t) + r_c^{ilm} \theta_c^i - \eta_p^{ilm} \tag{A7}$$

Appendix B: System Matrices

Many matrices have the same names as those in Lin and Parker [4]. Identically named matrices are similar or, in some cases identical, to those in [4]. Additionally, the $\mathbf{K}_{cp}$ and $\mathbf{K}_{gp}$ matrices are similar to the $\mathbf{K}_{c2}$, $\mathbf{K}_{r2}$, and $\mathbf{K}_{s2}$ matrices in [4].

$$\mathbf{x} = [\mathbf{x}_c^1 \ \cdots \ \mathbf{x}_c^a | \mathbf{x}_g^1 \ \cdots \ \mathbf{x}_g^b | \mathbf{x}_{ps}^1 \ \cdots \ \mathbf{x}_{ps}^a]^T$$

$$\mathbf{x}_c^i = [x_c^i \ y_c^i \ \theta_c^i]^T \quad \mathbf{x}_g^i = [x_g^j \ y_g^j \ \theta_g^j]^T \quad \mathbf{x}_{ps}^i = [\mathbf{x}_{pt}^{i1} \ \cdots \ \mathbf{x}_{pt}^{ic^i}]^T$$

$$\mathbf{x}_{pt}^{il} = [\mathbf{x}_p^{il1} \ \cdots \ \mathbf{x}_p^{ild^i}]^T \quad \mathbf{x}_p^{ilm} = [\zeta_p^{ilm} \ \eta_p^{ilm} \ u_p^{ilm}]^T$$

$$\mathbf{F}(t) = [\mathbf{F}_c^1(t) \ \cdots \ \mathbf{F}_c^a(t) \ \mathbf{F}_g^1(t) \ \cdots \ \mathbf{F}_g^b(t) \ 0 \ \cdots \ 0]^T$$

$$\mathbf{F}_c^i(t) = [F_{c,xx}^i(t) \ F_{c,yy}^i(t) \ \tau_c^i(t)]^T$$

$$\mathbf{F}_g^j(t) = [F_{g,xx}^j(t) \ F_{g,yy}^j(t) \ \tau_g^j(t)]^T$$

$$\mathbf{M} = diag(\mathbf{M}_c^1, \cdots \mathbf{M}_c^a, \mathbf{M}_g^1, \cdots \mathbf{M}_g^b, \mathbf{M}_{ps}^1, \cdots \mathbf{M}_{ps}^a)$$

$$\mathbf{M}_c^i = diag(m_c^i, m_c^i, I_c^i) \quad \mathbf{M}_g^j = diag(m_g^j, m_g^j, I_g^j)$$

$$\mathbf{M}_{ps}^i = diag(\mathbf{M}_{pt}^{i1}, \mathbf{M}_{pt}^{i2}, \cdots \mathbf{M}_{pt}^{ic^i})$$

$$\mathbf{M}_{pt}^{il} = diag(\mathbf{M}_p^{il11}, \mathbf{M}_p^{il22}, \cdots \mathbf{M}_p^{ild^i d^i})$$

$$\mathbf{M}_p^{ilmm} = diag(m_p^{ilm}, m_p^{ilm}, I_p^{ilm}/(r_p^{ilm})^2)$$

$$\mathbf{K} = \mathbf{K}_b + \mathbf{K}_m$$

$$\mathbf{K}_b = diag(\mathbf{K}_{cb}^1, \mathbf{K}_{cb}^2, \cdots \mathbf{K}_{cb}^a, \mathbf{K}_{gb}^1, \cdots \mathbf{K}_{gb}^b, \mathbf{0} \cdots \mathbf{0})$$

$$\mathbf{K}_{cb}^i = diag(k_{cb,xx}^i, k_{cb,yy}^i, K_{cb,\theta\theta}^i)$$

$$\mathbf{K}_{gb}^j = diag(k_{gb,xx}^j, k_{gb,yy}^j, K_{gb,\theta\theta}^j)$$

$$\mathbf{K}_m = \left[\begin{array}{ccc|ccc|ccc}
\mathbf{K}_c^{11} & \cdots & \mathbf{K}_c^{1a} & \mathbf{K}_{cg}^{11} & \cdots & \mathbf{K}_{cg}^{1b} & \mathbf{K}_{c,ps}^1 & & \\
 & \ddots & \vdots & \vdots & \ddots & \vdots & & \ddots & \\
 & & \mathbf{K}_c^{aa} & \mathbf{K}_{cg}^{a1} & \cdots & \mathbf{K}_{cg}^{ab} & & & \mathbf{K}_{c,ps}^a \\
\hline
 & & & \mathbf{K}_g^{11} & \cdots & \mathbf{K}_g^{1b} & \mathbf{K}_{g,ps}^{11} & \cdots & \mathbf{K}_{g,ps}^{1a} \\
 & & & & \ddots & \vdots & \vdots & \ddots & \vdots \\
 & & & & & \mathbf{K}_g^{bb} & \mathbf{K}_{g,ps}^{b1} & \cdots & \mathbf{K}_{g,ps}^{ba} \\
\hline
 & & & & & & \mathbf{K}_{ps}^1 & & \\
 & & & & & & & \ddots & \\
symmetric & & & & & & & & \mathbf{K}_{ps}^a
\end{array} \right]$$

$$\mathbf{K}_c^{in} = \begin{cases} \displaystyle\sum_{l=1}^{c^i} \sum_{m=1}^{d^i} \mathbf{K}_{c1}^{ilm} - \sum_{f=1,f\neq i}^{a} \mathbf{K}_{c4}^{if} - \sum_{j=1}^{b} \mathbf{K}_{cg}^{ij}, & \text{if } i = n \\ \mathbf{K}_{c4}^{in}, & \text{otherwise} \end{cases}$$

$$\mathbf{K}_{c1}^{ilm} = k_p^{ilm} \begin{bmatrix} 1 & 0 & -r_c^{ilm} \sin \psi^{ilm} \\ & 1 & r_c^{ilm} \cos \psi^{ilm} \\ symmetric & & (r_c^{ilm})^2 \end{bmatrix}$$

$$\mathbf{K}_{c4}^{in} = -\,diag(k_{c,xx}^{in}, k_{c,yy}^{in}, K_{c,\theta\theta}^{in})$$

$$\mathbf{K}_{cg}^{ij} = -\,diag(k_{cg,xx}^{ij}, k_{cg,yy}^{ij}, K_{cg,\theta\theta}^{ij})$$

$$\mathbf{K}_{c,ps}^{i} = [\mathbf{K}_{c,pt}^{i1}\ \mathbf{K}_{c,pt}^{i2}\ \cdots\ \mathbf{K}_{c,pt}^{ic^i}]$$

$$\mathbf{K}_{c,pt}^{il} = [\mathbf{K}_{cp}^{il1}\ \mathbf{K}_{cp}^{il2}\ \cdots\ \mathbf{K}_{cp}^{ild^i}]$$

$$\mathbf{K}_{cp}^{ilm} = k_p^{ilm}\begin{bmatrix} -\cos\psi^{ilm} & \sin\psi^{ilm} & 0 \\ -\sin\psi^{ilm} & -\cos\psi^{ilm} & 0 \\ 0 & -r_c^{ilm} & 0 \end{bmatrix}$$

$$\mathbf{K}_g^{jn} = \begin{cases} \displaystyle\sum_{i=1}^{a}\sum_{l=1}^{c^i}\sum_{m=1}^{d^i}\mathbf{K}_{g1}^{ilmn} - \sum_{f=1,f\neq j}^{b}\mathbf{K}_{g4}^{fj} - \sum_{i=1}^{a}(\mathbf{K}_{cg}^{ij})^T & \text{if } j = n \\ \mathbf{K}_{g4}^{jn}, & \text{otherwise} \end{cases}$$

$$\mathbf{K}_{g1}^{jilm} = \kappa_{gp}^{jilm}(t)\begin{bmatrix} \sin^2\psi_g^{jilm} & -\cos\psi_g^{jilm}\sin\psi_g^{jilm} & -r_g^j\sin\psi_g^{jilm} \\ & \cos^2\psi_g^{jilm} & r_g^j\cos\psi_g^{jilm} \\ symmetric & & (r_g^j)^2 \end{bmatrix}$$

$$\mathbf{K}_{g4}^{in} = -\,diag(k_{g,xx}^{in}, k_{g,yy}^{in}, K_{g,\theta\theta}^{in})$$

$$\mathbf{K}_{g,ps}^{ji} = [\mathbf{K}_{g,pt}^{ji1}\ \mathbf{K}_{g,pt}^{ji2}\ \cdots\ \mathbf{K}_{g,pt}^{jic^i}]$$

$$\mathbf{K}_{g,pt}^{jil} = [\mathbf{K}_{gp}^{jil1}\ \mathbf{K}_{gp}^{jil2}\ \cdots\ \mathbf{K}_{gp}^{jild^i}]$$

$$\mathbf{K}_{gp}^{jilm} = \kappa_{gp}^{jilm}(t) \times \begin{bmatrix} -\sigma^j\sin\psi_g^{jilm}\sin\alpha_g^{jilm} & \sin\psi_g^{jilm}\cos\alpha_g^{jilm} & \sigma^j\sin\psi_g^{jilm} \\ \sigma^j\cos\psi_g^{jilm}\sin\alpha_g^{jilm} & -\cos\psi_g^{jilm}\cos\alpha_g^{jilm} & -\sigma^j\cos\psi_g^{jilm} \\ r_g^j\sigma^j\sin\alpha_g^{jilm} & -r_g^j\cos\alpha_g^{jilm} & -r_g^j\sigma^j \end{bmatrix}$$

$$\mathbf{K}_{ps}^{i} = diag(\mathbf{K}_{pt}^{i1}, \mathbf{K}_{pt}^{i2}\cdots, \mathbf{K}_{pt}^{ic^i})$$

$$\mathbf{K}_{pt}^{il} = \begin{bmatrix} \mathbf{K}_p^{il11} & \cdots & \mathbf{K}_p^{il1d^i} \\ \vdots & \ddots & \vdots \\ \mathbf{K}_p^{ild^i1} & \cdots & \mathbf{K}_p^{ild^id^i} \end{bmatrix}$$

$$\mathbf{K}_p^{ilmn} = \begin{cases} \displaystyle\mathbf{K}_{c3}^{ilm} + \sum_{j=1}^{b}\mathbf{K}_{g3}^{jilm} + \sum_{f=1,f\neq m}^{d^i}\mathbf{K}_{p4}^{ilmf} & \text{if } m = n \\ \mathbf{K}_{p5}^{ilmn}, & \text{otherwise} \end{cases}$$

$$\mathbf{K}_{c3}^{ilm} = diag(k_p^{ilm}, k_p^{ilm}, 0)$$

$$\mathbf{K}_{g3}^{jilm} = \kappa_{gp}^{jilm}(t)\begin{bmatrix} \sin^2\alpha_g^{jilm} & -\sigma^j\cos\alpha_g^{jilm}\sin\alpha_g^{jilm} & -\sin\alpha_g^{jilm} \\ & \cos^2\alpha_g^{jilm} & \sigma^j\cos\alpha_g^{jilm} \\ symmetric & & 1 \end{bmatrix}$$

$$\mathbf{K}_{p4}^{ilmn} = \kappa_{pp}^{ilmn}(t)\begin{bmatrix} \sin^2\gamma^{ilmn} & -\cos\gamma^{ilmn}\sin\gamma^{ilmn} & -\sin\gamma^{ilmn} \\ & \cos^2\gamma^{ilmn} & \cos\gamma^{ilmn} \\ sym & & 1 \end{bmatrix} + diag\left(k_{pp,\zeta\zeta}^{ilmn}, k_{pp,\eta\eta}^{ilmn}, \frac{K_{pp,uu}^{ilmn}}{r_p^{ilm} r_p^{iln}}\right)$$

$$\mathbf{K}_{p5}^{ilmn} = \kappa_{pp}^{ilmn}(t) \times \begin{bmatrix} \sin\gamma^{ilmn}\sin\gamma^{ilnm} & -\sin\gamma^{ilmn}\cos\gamma^{ilnm} & -\sin\gamma^{ilmn} \\ -\cos\gamma^{ilmn}\sin\gamma^{ilnm} & \cos\gamma^{ilmn}\cos\gamma^{ilnm} & \cos\gamma^{ilmn} \\ -\sin\gamma^{ilnm} & \cos\gamma^{ilnm} & 1 \end{bmatrix} - diag\left(k_{pp,\zeta\zeta}^{ilmn}, k_{pp,\eta\eta}^{ilmn}, \frac{K_{pp,uu}^{ilmn}}{r_p^{ilm} r_p^{iln}}\right)$$

Note that if $\kappa_{pp}^{ilmn}(t) \neq 0$ then $k_{pp,\zeta\zeta}^{ilmn}=0, k_{pp,\eta\eta}^{ilmn}=0, K_{pp,uu}^{ilmn}=0$. If $k_{pp,\zeta\zeta}^{ilmn} \neq 0$, $k_{pp,\eta\eta}^{ilmn} \neq 0$, or $K_{pp,uu}^{ilmn} \neq 0$ then $\kappa_{pp}^{ilmn}(t)=0$. This is because two planets cannot be meshed with each other and shaft connected with each other (i.e., stepped) at the same time.

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